Spring 2025, Math 223D, Homework 1. Recommended due date: Apr 14.

Problem 1. Let X be an arbitrary set. Show that for every subset $A \subseteq X$, its characteristic function $\mathbf{1}_A \colon X \to 2$ is computed by some Boolean circuit (not necessarily countable).

Problem 2. Show that every countable set is Borel.

Problem 3. Show that the following sets are Borel in the space Graphs of all graphs with vertex set $\mathbb{N}$ (identified with the Cantor space $\mathcal{C}$ as discussed in class).

(a) The set of all connected graphs.

(b) The set of all trees.

(c) The set of all bipartite graphs.

(d) The set of all locally finite graphs (i.e., graphs where every vertex has finitely many neighbors).

(e) The set of all k-colorable graphs for fixed $k \in \mathbb{N}$. (This is a bit trickier.)

Problem 4. For $x \in \mathcal{C}$ and $n \geqslant 1$, let $A_n(x) := \sum_{i=0}^{n-1} x_i/n$ be the average of the first n bits of x. Call a point $x \in \mathcal{C}$ *erratic* if for every $\delta \in [0,1]$, the sequence $(A_n(x))_{n \geqslant 1}$ has a subsequence that converges to δ. (So an erratic point $x \in \mathcal{C}$ is as far from having a well-defined density as possible.) Show that the set of all erratic points is a Borel subset of $\mathcal{C}$.

Problem 5. Prove the fact, stated in class, that the family $\boldsymbol{B}(\mathcal{C})$ of all Borel subsets of $\mathcal{C}$ (defined using countable Boolean circuits) is the smallest σ-algebra on $\mathcal{C}$ that includes every open set.

Problem 6.

(a) Prove that every clopen subset of $\mathcal{C}$ is finitely determined.

(b) Give an example of a clopen set in $\mathcal{N} := \mathbb{N}^{\mathbb{N}}$ that is not finitely determined.

Problem 7. Show that the set of all sequences $x \in \mathcal{C}$ with finitely many 1s is countable and dense.

Problem 8. Let $\mathbb{Q}_2$ be the set of all *dyadic rationals*, i.e., rational numbers whose denominator in lowest terms is a power of 2. Prove that $(0,1)\backslash\mathbb{Q}_2$ is homeomorphic to the set of all sequences $x \in \mathcal{C}$ with infinitely many 0s and 1s.

Problem 9. Show that $\mathcal{N}$ is homeomorphic to the set of all sequences $x \in \mathcal{C}$ with infinitely many 1s.

Problem 10. Show that the spaces $2^{\mathbb{N}}$ and $3^{\mathbb{N}}$ are homeomorphic.