**Spring 2025, Math 223D, Homework 1.** Recommended due date: Apr 21.

**Problem 1.** Prove that the normalized Hamming metric on $\mathcal{C}$ is complete.

**Problem 2.** Let $S_\infty$ be the group of all permutations of $\mathbb{N}$, i.e., all bijections $\mathbb{N} \to \mathbb{N}$. View $S_\infty$ as a subset of $\mathcal{N} = \mathbb{N}^\mathbb{N}$ and endow it with the subspace topology. Show that $S_\infty$ is a Polish space.

**Problem 3.** Let $X$ be a standard Borel space and let $f\colon X \to X$ be a Borel function. Show that there is a compatible Polish topology $\tau$ on $X$ such that $f\colon (X,\tau) \to (X,\tau)$ is continuous.

**Problem 4.** Let $X$ be a standard Borel space and let $\tau$ be some Polish topology on $X$ such that every $\tau$-open set is Borel. Show that $\tau$ is compatible (i.e., $\boldsymbol{B}(X)$ is the Borel $\sigma$-algebra of $(X,\tau)$).

**Problem 5.** Show that the set

$$\mathsf{GraphIso} \coloneqq \{(G,H) \in \mathsf{Graphs} \times \mathsf{Graphs} \,:\, G \text{ and } H \text{ are isomorphic}\},$$

is analytic. (Here $\mathsf{Graphs}$ is the space of all graphs with vertex set $\mathbb{N}$.)

**Problem 6.** Given a directed graph $D \in \mathsf{Dir}$, define a corresponding undirected graph $F_D$ as follows. The vertex set of $F_D$ is $\mathbb{N}^* \setminus \{\varnothing\}$, i.e., the set of all nonempty finite sequences of natural numbers. We put an (undirected) edge in $F_D$ between sequences $v = (v_1, \ldots, v_k)$ and $u = (u_1, \ldots, u_\ell)$, where $k \leqslant \ell$, if and only if $\ell = k+1$, $u_i = v_i$ for all $1 \leqslant i \leqslant k$, and $(u_\ell, v_k) \in E(D)$.

(i) Prove that the graph $F_D$ is acyclic, i.e., a forest.

(ii) Show that $F_D$ contains an infinite path if and only if $D$ is not well-founded.

Conclude that the sets

$$\mathsf{Path}_\infty \coloneqq \{G \in \mathsf{Graphs} \,:\, G \text{ contains an infinite path}\};$$
$$\mathsf{ForestPath}_\infty \coloneqq \{F \in \mathsf{Graphs} \,:\, F \text{ is a forest with an infinite path}\}$$

are complete analytic, hence not Borel.

**Problem 7.** Prove that the sets

$$\mathsf{TreePath}_\infty \coloneqq \{T \in \mathsf{Graphs} \,:\, T \text{ is a tree with an infinite path}\};$$
$$\mathsf{Clique}_\infty \coloneqq \{G \in \mathsf{Graphs} \,:\, G \text{ has an infinite clique}\}$$

are complete analytic, hence not Borel.

**Problem 8.** Recall that a graph $G$ is called *locally finite* if every vertex of $G$ has finitely many neighbors. Show that the set of all locally finite graphs with an infinite path is Borel.

**Problem 9.** Let $C$ be a Boolean pseudo-circuit in which negations are only applied at the inputs (meaning that for every `not`-gate $v$, the unique in-neighbor of $v$ is an input node). Show that $C$ admits at least one evaluation on every input (but possibly more than one if $C$ is not well-founded).

**Problem 10.** In this problem we use the assignment of Boolean pseudo-circuits $C_x$ to the points $x \in \mathcal{C}$ of the Cantor space discussed in class. Recall that the pseudo-circuits $C_x$ for $x \in \mathcal{C}$ have the same countable vertex set, which we denote by $V$, and the same output node, which we denote by `out`. Let $\mathsf{Z}_\forall$ be the set of all points $x \in \mathcal{C}$ such that:

*for every evaluation* $\mathsf{eval}\colon V \to \{0,1\}$ *of* $C_x$ *on the input* $x$, $\mathsf{eval}(\mathtt{out}) = 0$.

Also, let $\mathsf{U}_\forall$ be the set of all $x \in \mathcal{C}$ such that:

*for every evaluation* $\mathsf{eval}\colon V \to \{0,1\}$ *of* $C_x$ *on the input* $x$, $\mathsf{eval}(\mathtt{out}) = 1$.

Prove that $\mathsf{Z}_\forall$ and $\mathsf{U}_\forall$ are disjoint co-analytic subsets of $\mathcal{C}$ that cannot be separated by a Borel set, i.e., there is no partition $\mathcal{C} = B_0 \sqcup B_1$ of $\mathcal{C}$ into two Borel sets such that $\mathsf{Z}_\forall \subseteq B_0$ and $\mathsf{U}_\forall \subseteq B_1$.

**Hint.** To prove that $\mathsf{Z}_\forall$ and $\mathsf{U}_\forall$ are disjoint, you will need to use Problem 9.