Spring 2025, Math 223D, Homework 3. Recommended due date: Apr 27.

Problem 1. A *standard measure space* is a pair (X, μ), where X is a standard Borel space and μ is a measure on X. Show that the following statements are equivalent for a function $f \colon X \to Y$ from a standard measure space (X, μ) to a standard Borel space Y:

(i) there exists a Borel function $g \colon X \to Y$ such that the set $\{x \in X \,:\, f(x) = g(x)\}$ is μ-conull,

(ii) the preimage of every Borel subset of Y under f is μ-measurable.

A function f satisfying these equivalent conditions is called *μ-measurable*.

Problem 2. State and prove the version of Problem 1 for Baire-measurable functions.

Problem 3. Is it true that for every Borel set $A \subseteq \mathbb{R}$, there is an open set $U \subseteq \mathbb{R}$ with $A \triangle U$ null?

Problem 4. Prove that if μ is a finite measure on $\mathcal{C}$ and $A \subseteq \mathcal{C}$ is a μ-measurable set such that $\mu(A) > 0$, then for every $\varepsilon > 0$, there is a non-null finitely determined set $U \subseteq \mathcal{C}$ with

$$\frac{\mu(A \triangle U)}{\mu(U)} \leqslant \varepsilon.$$

Problem 5. Let X be a Polish space without isolated points and let μ be an atomless measure on X (*atomless* means that every singleton is μ-null). Show that X has a μ-null comeager subset.

Problem 6. For each of the following subsets of Graphs, determine whether it is meager, comeager, or neither, as well as whether it is null, conull, or neither with respect to the fair coin-flip measure:

(i) $\mathsf{Bip} \coloneqq \{G \in \mathsf{Graphs} \,:\, G \text{ is bipartite}\}$,

(ii) $\mathsf{Conn} \coloneqq \{G \in \mathsf{Graphs} \,:\, G \text{ is connected}\}$,

(iii) $\mathsf{Mat} \coloneqq \{G \in \mathsf{Graphs} \,:\, G \text{ has a perfect matching}\}$.

Problem 7. Call a graph $G \in \mathsf{Graphs}$ *saturated* if for every pair of disjoint finite sets $S, T \subseteq \mathbb{N}$, there is a vertex $u \in \mathbb{N} \setminus (S \cup T)$ that is adjacent to every vertex in S and to no vertex in T.

(i) Show that the set $\mathsf{Sat} \coloneqq \{G \in \mathsf{Graphs} \,:\, G \text{ is saturated}\}$ is comeager and conull with respect to the fair coin-flip measure.

(ii) Show that every two saturated graphs in Graphs are isomorphic.

Hint. Take two saturated graphs G, $H \in \mathsf{Graphs}$. Construct an isomorphism $f \colon \mathbb{N} \to \mathbb{N}$ between G and H inductively over the course of countably many steps. On step i, decide where to map the i-th vertex of G and which vertex is mapped to the i-th vertex of H.

(iii) Conclude that there exists a graph $\mathbf{R} \in \mathsf{Graphs}$, called the *Rado graph*, such that the set $\{G \in \mathsf{Graphs} \,:\, G \cong \mathbf{R}\}$ is comeager and conull (with respect to the fair coin-flip measure).

Problem 8. Fix a set $A \subseteq \mathcal{C}$. Alice and Bob are playing the following game, called the *Banach–Mazur game* $\mathtt{BM}(A)$. Alice takes the first turn. On their turn, each player picks an arbitrary nonempty finite sequence of 0s and 1s. This produces sequences s_0, s_1, s_2, ... as shown below:

Alice	s_0		s_2		...	s_{2k}		...
Bob		s_1		s_3	...		s_{2k+1}	...

The game continues indefinitely and yields an infinite string $x := s_0 {}^\frown s_1 {}^\frown s_2 {}^\frown \cdots$. Alice wins if and only if $x \in A$. Show that Bob has a winning strategy in this game if and only if A is meager.

Caution. It may happen that neither Alice nor Bob has a winning strategy—see the next problem!

Problem 9. Recall the infinite dimensional hypercube graph $\mathbb{H}$. As discussed in class, $\mathbb{H}$ is bipartite. Show that if $\mathcal{C} = A \sqcup B$ is a partition of the Cantor space into two $\mathbb{H}$-independent sets, then neither Alice nor Bob has a winning strategy in the Banach–Mazur game $\mathtt{BM}(A)$.

Problem 10. Let $D \subseteq \mathbb{R}$ and define a graph G_D with vertex set $\mathbb{R}$ and edge set $\{\{x, y\} : |x-y| \in D\}$. Show that $\chi_{\mathsf{B}}(G_D) \leqslant \aleph_0$ if and only if 0 is not in the closure of D.