Spring 2025, Math 223D, Homework 4. Recommended due date: May 6.

Problem 1. Let X be a Polish space and let $T\colon X \to X$ be a Borel map without fixed points. Fill in the details of the proof that $\chi_{\mathsf{BM}}(G_T) \leqslant 3$ sketched in class.

Problem 2. Recall that $\mathcal{N} = \mathbb{N}^{\mathbb{N}}$. Let $D \subseteq \mathcal{N}$ be the set of all sequences $(n_i)_{i \in \mathbb{N}}$ such that $n_i \neq n_{i+1}$ for all $i \in \mathbb{N}$. Define a map $F\colon D \to D$ by

$$F\big((n_0, n_1, n_2, \ldots)\big) \coloneqq (n_1, n_2, n_3, \ldots).$$

Note that, by the definition of D, F has no fixed points. Show that if X is a standard Borel space and $T\colon X \to X$ is a Borel map without fixed points, then there exists a Borel homomorphism $\varphi\colon X \to D$ from G_T to G_F. (A *homomorphism* from a graph G to a graph H is a function $\varphi\colon V(G) \to V(H)$ that sends edges of G to edges of H, i.e., if $\{u, v\} \in E(G)$, then $\{\varphi(u), \varphi(v)\} \in E(H)$.)

Problem 3. Let X be a standard Borel space and let $T\colon X \to X$ be a Borel map without fixed points. Show that there exists a Borel proper coloring $c\colon X \to \mathbb{N}$ of G_T such that for all $x \in X$, we have $c(x) < c(Tx)$ unless $c(x) = 2$.

Problem 4. Let X be a standard Borel space and let T_1, $T_2\colon X \dashrightarrow X$ be Borel partial maps (i.e., for each $i \in \{1, 2\}$, $\mathrm{dom}(T_i) \subseteq X$ is a Borel set and $T_i\colon \mathrm{dom}(T_i) \to X$ is a Borel function). Assume T_1, T_2 have no fixed points. Define the graph G_{T_1,T_2} with vertex set X and edge set

$$\big\{\{x, T_1x\} \,:\, x \in \mathrm{dom}(T_1)\big\} \,\cup\, \big\{\{x, T_2x\} \,:\, x \in \mathrm{dom}(T_2)\big\}.$$

Suppose that the Borel chromatic number $\chi_{\mathsf{B}}(G_{T_1,T_2})$ is finite and the set $\mathrm{dom}(T_1) \cap \mathrm{dom}(T_2)$ is independent in G_{T_1,T_2}. Show that $\chi_{\mathsf{B}}(G_{T_1,T_2}) \leqslant 3$.

Hint. For $x \in \mathrm{dom}(T_1) \,\triangle\, \mathrm{dom}(T_2)$, let $Tx \coloneqq T_ix$, where $i \in \{1, 2\}$ is the unique index such that $x \in \mathrm{dom}(T_i)$. Consider two cases depending on whether there exists some $n \in \mathbb{N}$ such that $T^n x \in \mathrm{dom}(T_1) \cap \mathrm{dom}(T_2)$.

Problem 5. In this problem you will prove the following:

Theorem (Palamourdas)**.** *Let X be a standard Borel space and let T_1, $T_2\colon X \to X$ be Borel maps without fixed points. Let $G_{T_1,T_2} \coloneqq G_{T_1} \cup G_{T_2}$. If $\chi_{\mathsf{B}}(G_{T_1,T_2}) < \aleph_0$, then $\chi_{\mathsf{B}}(G_{T_1,T_2}) \leqslant 5$.*

Suppose we are in the setting of the theorem and let $f\colon X \to q$ be a Borel proper coloring of G_{T_1,T_2} using finitely many colors. Define functions c, $c'\colon X \to \{0, 1, 2\}$ recursively by

$$c(x) \coloneqq \min\big\{k \,:\, c(T_ix) \neq k \text{ for all } i \in \{1, 2\} \text{ such that } f(T_ix) > f(x)\big\},$$
$$c'(x) \coloneqq \min\big\{k \,:\, c'(T_ix) \neq k \text{ for all } i \in \{1, 2\} \text{ such that } c(T_ix) = c(x)\big\}.$$

We will discuss this definition in class on April 30. In particular, we will see that (c, c') is a Borel proper coloring of G_{T_1,T_2} with 6 colors (i.e., one more than stated in the theorem).

Let $Y \subseteq X$ be the set of all points $x \in X$ such that

$$(c(x), c'(x)) \,\in\, \big\{(0, 1),\, (0, 2),\, (1, 1),\, (2, 0)\big\}.$$

(i) Show that the set $Y \cap T_1^{-1}(Y) \cap T_2^{-1}(Y)$ is independent in G_{T_1,T_2}.

(ii) Use Problem 4 to deduce that the induced subgraph $G_{T_1,T_2}[Y]$ has $\chi_{\mathsf{B}}(G_{T_1,T_2}[Y]) \leqslant 3$.

(iii) Finish the proof of the theorem.

Problem 6. Let X be a Polish space and let $(G_n)_{n \in \mathbb{N}}$ be locally countable graphs with vertex set X such that $\chi_{\mathsf{B}}(G_n) \leqslant \aleph_0$ for all $n \in \mathbb{N}$. Show that there exists a partition $X = \bigsqcup_{n \in \mathbb{N}} X_n$ such that each X_n is Baire-measurable and G_n-independent.

Hint. Take a countable basis $(U_n)_{n \in \mathbb{N}}$ for the topology on X and pick X_n so that $X_n \cap U_n$ is not meager.

Problem 7. Finish the proof that Ellentuck-open sets are *completely* Ramsey. (We proved in class that they are Ramsey.)

Problem 8. A set $\mathcal{U} \subseteq [\mathbb{N}]^\infty$ is called *Ramsey null* if and for all $a \in [\mathbb{N}]^{<\infty}$ and $A \in [\mathbb{N}]^\infty$, there exists $X \in [A]^\infty$ such that $[a, X] \cap \mathcal{U} = \varnothing$. Show that $\mathcal{U}$ is Ramsey null if and only if it is nowhere dense (equivalently, meager) in the Ellentuck topology.

Problem 9. Here we equip $[\mathbb{N}]^\infty$ with the ordinary Polish topology obtained by identifying it with $\{x \in \mathcal{C} : x \text{ contains infinitely many 1s}\}$. Let X be a Polish space and let $f \colon [\mathbb{N}]^\infty \to X$ be a Borel function. Show that for some $S \in [\mathbb{N}]^\infty$, the restriction of f to $[S]^\infty$ is continuous.

Hint. Iterate the following operation: Given $A \in [\mathbb{N}]^\infty$, $a \in [\mathbb{N}]^{<\infty}$, and an open set $U \subseteq X$, find $A' \in [A]^\infty$ such that for all $B \in [A' \cup a]^\infty$, the intersection $B \cap a$ determines whether $f(B) \in U$.