Spring 2025, Math 223D, Homework 5. Recommended due date: May 13.

Problem 1. Let X be a Polish space. Show that there is a linear order $<$ on X that is simultaneously G_δ and F_σ (in the product topology on X^2).

Hint. Use a countable basis for the topology on X to define an injective map $X \to \mathcal{C}$.

Problem 2. Let X be a standard Borel space and let $k \in \mathbb{N}$. In class, we introduced the following three equivalent ways of endowing $[X]^k$ with a σ-algebra of Borel sets:

(I) Let $(X)^k \subseteq X^k$ be the set of all k-tuples of distinct elements of X and define

$$\pi\colon (X)^k \to [X]^k \colon (x_1, \ldots, x_k) \mapsto \{x_1, \ldots, x_k\}.$$

Then declare a set $A \subseteq [X]^k$ to be Borel if and only if $\pi^{-1}(A)$ is a Borel subset of $(X)^k$.

(II) Let d be a compatible metric on X and define a metric $[d]^k$ on $[X]^k$ by

$$[d]^k\big(\{x_1, \ldots, x_k\}, \{y_1, \ldots, y_k\}\big) := \min_{\sigma} \max_{1 \leqslant i \leqslant k} d\left(x_i, y_{\sigma(i)}\right),$$

where the minimum is taken over all permutations $\sigma\colon \{1, \ldots, k\} \to \{1, \ldots, k\}$. Now a set $A \subseteq [X]^k$ is Borel if and only if it is Borel in the metric space $([X]^k, [d]^k)$.

(III) Let $<$ be a Borel linear order on X and make a set $A \subseteq [X]^k$ Borel if and only if the set

$$\big\{(x_1, \ldots, x_k) \in X^k \,:\, \{x_1, \ldots, x_k\} \in A \text{ and } x_1 < \cdots < x_k\big\}$$

is Borel in X^k. (That is, we identify $[X]^k$ with the set of all increasing k-tuples of elements of X.)

Verify that these three definitions indeed give rise to the same Borel σ-algebra on $[X]^k$.

Problem 3. Let X be a standard Borel space and let $k, \ell \in \mathbb{N}$. Show that the following set is Borel:

$$\{(a, b) \in [X]^k \times [X]^\ell \,:\, a \subseteq b\}.$$

Problem 4. Let $S \subseteq 2^*$. Show that:

(i) if S contains at most one string of each finite length, then $\mathbb{G}_S$ has no cycles,

(ii) if S contains at least one string of each finite length, then two vertices $x, y \in \mathcal{C}$ belong to the same component of $\mathbb{G}_S$ if and only if they differ in finitely many coordinates.

Problem 5. Let $\mathbb{S} := \prod_{n \in \mathbb{N}} \{0,1\}^n \cong \mathcal{C}$ be the space of all sequences of the form $(s_n)_{n \in \mathbb{N}}$, where for each $n \in \mathbb{N}$, s_n is a bit string of length n, and let $\mathbb{D} \subseteq \mathbb{S}$ be the set of all sequences $(s_n)_{n \in \mathbb{N}}$ such that the set $\{s_n \,:\, n \in \mathbb{N}\}$ of finite bit strings is dense. Show that $\mathbb{D}$ is comeager in $\mathbb{S}$.

Problem 6. Let X, I be standard Borel spaces. We say that $(G_i)_{i \in I}$ is a *Borel family of graphs* on X indexed by I if each G_i is a graph on X and the set

$$\{(i, e) \in I \times [X]^2 \,:\, e \in E(G_i)\}$$

is Borel. A *Borel family of countable colorings* for $(G_i)_{i \in I}$ is a collection $(f_i)_{i \in I}$, where each $f_i\colon X \to \mathbb{N}$ is a proper coloring of the corresponding graph G_i and the map

$$I \times X \to \mathbb{N}\colon (i, x) \mapsto f_i(x)$$

is Borel. Show that the following statements are equivalent for a Borel family of graphs $(G_i)_{i \in I}$:

(i) $\chi_{\mathsf{B}}(G_i) \leqslant \aleph_0$ for all $i \in I$,

(ii) there exists a Borel family of countable colorings $(f_i)_{i \in I}$ for $(G_i)_{i \in I}$.

Problem 7. Give an example of a Borel graph G such that the set $\{x \in V(G) : \deg_G(x) = 0\}$ of all isolated vertices in G is not Borel.

Hint. Start by fixing an arbitrary non-Borel coanalytic set.

Problem 8. A set $U \subseteq V(G)$ in a graph G is called G-*invariant* if G contains no edges joining U to U^{c}, i.e., if U is a union of connected components of G. Suppose G is an analytic graph and A, $B \subseteq V(G)$ are analytic sets such that there is no path in G from A to B. Show that there exists a Borel G-invariant set $U \subseteq V(G)$ with $U \supseteq A$ and $U \cap B = \varnothing$.

Problem 9. Let μ be the fair coin-flip measure on $\mathcal{C}$ and let $S \subset 2^*$ be a set that includes exactly one string of each finite length.

(i) Show that μ-almost every vertex $x \in \mathcal{C}$ has finite degree in the graph $\mathbb{G}_S$.

(ii) Conclude that $\chi_\mu(\mathbb{G}_S) \leqslant \aleph_0$.

(iii) Show that $\chi_\mu(\mathbb{G}_S) > 2$ (even though $\mathbb{G}_S$ is acyclic).

Remark. In fact, $\chi_\mu(\mathbb{G}_S) = 3$. You can try proving this as a bonus!