Spring 2025, Math 223D, Homework 6. Recommended due date: May 22.

Problem 1. Let X, Y be standard Borel spaces and let $A \subseteq X \times Y$ be a Borel set such that for all $x \in X$, the fiber A_x is countable. Recall that, by the Luzin–Novikov theorem, there exist Borel partial maps $(f_n \colon X \dashrightarrow Y)_{n \in \mathbb{N}}$ such that

$$A = \{(x, y) \in X \times Y \,:\, y = f_n(x) \text{ for some } n \in \mathbb{N}\}.$$

(i) Show that if $A_x \neq \varnothing$ for every point $x \in X$, then it is possible to arrange that each f_n is defined on all of X.

(ii) Show that if A_x is infinite for all $x \in X$, then it is possible to arrange that each f_n is defined on all of X and for every point $x \in X$, the map $\mathbb{N} \to A_x \colon n \mapsto f_n(x)$ is a bijection.

Problem 2. Let G be a locally countable Borel graph. Show that $\deg_G \colon V(G) \to \mathbb{N} \cup \{\aleph_0\}$ is a Borel function.

Problem 3. Let G be a locally countable Borel graph. Show that the set of all vertices $v \in V(G)$ that are contained in an odd cycle in G is Borel.

Problem 4. Let G be a graph of maximum degree $d \in \mathbb{N}$ and without isolated vertices. Show that there exist Borel maps $T_1, \ldots, T_d \colon V(G) \to V(G)$ with no fixed points such that $G = G_{T_1,\ldots,T_d}$.

Problem 5. Let X be a standard Borel space and let $[X]^{<\infty} := \bigcup_{k \in \mathbb{N}} [X]^k$. Let $\mathcal{H} \subseteq [X]^{<\infty}$ be a Borel set. The *intersection graph* of $\mathcal{H}$ is the graph $G_{\mathcal{H}}$ with vertex set $\mathcal{H}$ and edge set

$$\big\{\{h, h'\} \in [\mathcal{H}]^2 \,:\, h \cap h' \neq \varnothing\big\}.$$

Show that $\chi_{\mathsf{B}}(G_{\mathcal{H}}) \leqslant \aleph_0$ if and only if $G_{\mathcal{H}}$ is locally countable.

Hint. Generalize the proof of the Feldman–Moore theorem.

Problem 6. Follow the following steps to complete the proof of the injective form of the $\mathbb{G}_0$-dichotomy for locally countable graphs sketched in class. See Lectures 16 and 20 for the definitions of the terms used below (as well as Lectures 17 and 18 for the necessary background results).

Let G be a locally countable Borel graph and fix a Borel proper edge-coloring $c \colon E(G) \to \mathbb{N}$ of G.

(i) Show that if $\mathcal{H} \subseteq \mathsf{Hom}(H, G)$ is a large injective Borel set, there there is a large strongly injective Borel subset $\mathcal{H}' \subseteq \mathcal{H}$.

(ii) Show that if $\mathcal{H} \subseteq \mathsf{Hom}(H, G)$ is a large Borel set, there there is a large consistent Borel subset $\mathcal{H}' \subseteq \mathcal{H}$.

Now, let $S = \{s_n \,:\, n \in \mathbb{N}\}$, where for each $n \in \mathbb{N}$, s_n is a binary string of length n. Define the graphs H_n as in the proof of the $\mathbb{G}_0$-dichotomy, so $H_0 \cong K_1$ and $H_{n+1} = H_n +_{s_n} H_n$ for all $n \in \mathbb{N}$.

(iii) Construct a sequence of Borel sets $\mathcal{H}_n \subseteq \mathsf{Hom}(H_n, G)$ so that:

- $\mathcal{H}_n$ is large, consistent, and strongly injective,
- $\mathcal{H}_{n+1} \subseteq \mathcal{H}_n +_{s_n} \mathcal{H}_n$,
- for each $x \in V(H_n) \cup E(H_n)$, $\mathsf{diam}(\mathcal{H}_n(x)) \leqslant 2^{-n}$.

(iv) Use the sequence $(\mathcal{H}_n)_{n \in \mathbb{N}}$ to define an injective continuous homomorphism from $\mathbb{G}_S$ to G.

Problem 7. Suppose G is a Borel forest (i.e., a graph with no cycles) on a Polish space X. Let $S \subset 2^*$ be a set containing exactly one binary string of each finite length. Show that if $\chi_{\mathsf{B}}(G) = 2^{\aleph_0}$, then there exists an injective continuous homomorphism from $\mathbb{G}_S$ to G.

Caution. Note that G is not assumed to be locally countable!

Hint. Adapt the proof of the $\mathbb{G}_0$-dichotomy using strongly injective families $\mathcal{H} \subseteq \mathsf{Hom}(H, G)$.

Problem 8.

(i) Let X be a standard Borel space and let $Y \subseteq X$ be an uncountable Borel subset. Show that if τ_Y is a compatible Polish topology on Y, then there exists a compatible Polish topology τ_X on X such that a set $A \subseteq X$ is τ_X-meager if and only if $A \cap Y$ is τ_Y-meager.

Hint. Pick an uncountable τ_Y-meager Borel set $M \subseteq Y$ and invoke a Borel isomorphism $M \cup A^{\mathsf{c}} \cong M$.

(ii) Use part (i) to show that a locally countable Borel graph G has uncountable Borel chromatic number if and only if there exists a compatible Polish topology τ on $V(G)$ with respect to which every Baire-measurable G-independent set is meager.

Hint. Apply the injective form of the $\mathbb{G}_0$-dichotomy.

Problem 9. Recall the ∞-dimensional hypercube graph $\mathbb{H}$. We say that a function $f \colon \mathcal{C} \to Y$ is $\mathbb{H}$-*invariant* if it is constant on the connected components of $\mathbb{H}$. We say that $f \colon \mathcal{C} \to Y$ is *generically constant* if f is constant on some comeager set $X \subseteq \mathcal{C}$.

(i) Show that every $\mathbb{H}$-invariant Baire-measurable function $f \colon \mathcal{C} \to \{0, 1\}$ is generically constant.

(ii) Show that every $\mathbb{H}$-invariant Baire-measurable function $f \colon \mathcal{C} \to \mathcal{C}$ is generically constant.

(iii) Show that for any standard Borel space X, every $\mathbb{H}$-invariant Baire-measurable function $f \colon \mathcal{C} \to X$ is generically constant.