Spring 2025, Math 223D, Homework 7. Recommended due date: June 4.

Problem 1. Fix $\alpha \in \mathbb{R}\backslash\pi\mathbb{Q}$ and let $T_\alpha\colon \mathbb{S}^1 \to \mathbb{S}^1$ be the rotation by α. Let $G := G_{T_\alpha}$. We use μ to denote the Lebesgue measure on $\mathbb{S}^1$ normalized so that $\mu(\mathbb{S}^1) = 1$. Recall that $\chi_{\mathsf{B}}(G) = \chi_\mu(G) = 3$.

(i) Show that for every $\varepsilon > 0$, G has a Borel proper 3-coloring in which the measure of one of the color classes is less than ε.

Hint. Take a tiny interval $I \subset \mathbb{S}^1$ and argue that $\chi_{\mathsf{B}}(G - I) = 2$.

(ii) Show that G has a Borel proper 3-coloring in which every color class has measure 1/3.

Caution. This is a somewhat trickier problem.

Problem 2. We continue working in the setting of Problem 1.

(i) Show that if $I \subseteq \mathbb{S}^1$ is a measurable G-independent set, then $\mu(I) < 1/2$.

(ii) Show that for every $\varepsilon > 0$, there is a measurable G-independent set I with $\mu(I) > 1/2 - \varepsilon$.

Problem 3. We continue working in the setting of Problems 1 and 2. Show that T_α is an *ergodic* transformation, meaning that if $A \subseteq \mathbb{S}^1$ is a measurable T_α-invariant set, then $\mu(A) \in \{0, 1\}$.

Problem 4. Call a binary sequence $x = (x_n)_{n\in\mathbb{Z}} \in \{0,1\}^{\mathbb{Z}}$ indexed by the integers *periodic* if there is some $t \in \mathbb{Z}\backslash\{0\}$ such that $x_n = x_{n+t}$ for all $n \in \mathbb{Z}$, and *aperiodic* otherwise. Let $\mathcal{A} \subset \{0,1\}^{\mathbb{Z}}$ be the set of all aperiodic sequences and define $T\colon \mathcal{A} \to \mathcal{A}$ by

$$T\big((x_n)_{n\in\mathbb{Z}}\big) := (x_{n+1})_{n\in\mathbb{Z}}.$$

In other words, the transformation T "shifts" the given sequence to the left by one position. Show that the graph G_T satisfies $\chi(G_T) = 2$ but $\chi_{\mathsf{B}}(G_T) = 3$.

Problem 5. A *matching* in a graph is a set of pairwise disjoint edges. Show that every locally countable Borel graph has a Borel maximal matching.

Problem 6. A matching in a graph is *perfect* if it covers every vertex. Working in the setting of Problems 1–3, show that the graph $G = G_{T_\alpha}$ has a perfect matching but no Borel perfect matching.

Problem 7. Show that if a component-finite Borel graph has a perfect matching, then it has a Borel perfect matching as well.

Problem 8. Let G be a component-finite Borel graph and let $L\colon V(G) \to [\mathbb{N}]^{<\infty}$ be a Borel function. Show that if G has a proper L-coloring (i.e., a proper coloring $c\colon V(G) \to \mathbb{N}$ with $c(v) \in L(v)$ for all $v \in V(G)$), then it has a Borel proper L-coloring as well.

Remark. This fact is used in Lecture 25 (on 05/28/25).

Problem 9. Let G be a locally finite Borel graph and let $L\colon V(G) \to [\mathbb{N}]^{<\infty}$ be a Borel function.

(i) Show that if $|L(v)| > \deg_G(v)$ for all $v \in V(G)$, then G has a Borel proper L-coloring.

(ii) Furthermore, show that if $|L(v)| \geqslant \deg_G(v)$ for all $v \in V(G)$ and every component of G contains at least one vertex x with $|L(x)| > \deg_G(x)$, then G has a Borel proper L-coloring.

Problem 10. Let G be a locally finite Borel graph with vertex set V and edge set E. Show that there is a partition $E = C \sqcup T$ of E into Borel sets C and T with the following properties:

- every vertex in the graph (V, C) has even degree, and
- the graph (V, T) is a forest.

Problem 11. Let G be a locally countable Borel graph and suppose that there exists a *Borel transversal* for G, i.e., a Borel set $T \subseteq V(G)$ that meets every component of G in exactly one vertex. The goal of this problem is to prove that $\chi_{\mathsf{B}}(G) = \chi(G)$.

(i) Argue that we may assume every component of G is infinite.

(ii) Assuming every component of G is infinite, construct a partition $V(G) = \bigcup_{n\in\mathbb{N}} T_n$ of the vertex set of G into Borel transversals.

(iii) Conclude that $\chi_{\mathsf{B}}(G) \leqslant \aleph_0$.

(iv) Let $k := \chi(G)$. Inductively define Borel k-colorings $c_n\colon T_0 \cup \ldots \cup T_n \to k$ so that c_n can be extended to a (not necessarily Borel) proper k-coloring of G.

Remark. You may use the fact that for $k \in \mathbb{N}$, a proper partial k-coloring can be extended to a proper k-coloring of the whole graph if and only if it can be extended to every finite subgraph.

(v) Conclude that $\chi_{\mathsf{B}}(G) \leqslant k$ and thus $\chi_{\mathsf{B}}(G) = \chi(G)$, as desired.