Math 223D, Lecture 1

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Syllabus on BruinLearn

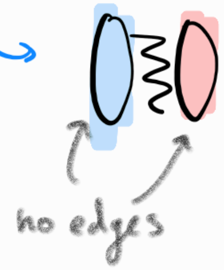
Descriptive Combinatorics

Descriptive → descriptive set theory

Combinatorics → graph theory, etc.

Seek constructive ? solutions to combinatorial problems.

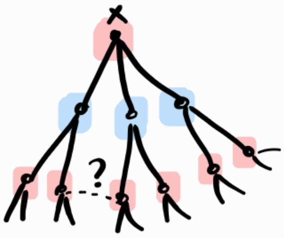
Fact: A graph with no odd cycles is bipartite, i.e., 2-colorable.



$\chi(G) \leq 2$ — chromatic #

Pf: Start with any vtx x.

Color x red.
Color the nbrs of x blue.
Color their nbrs red, etc.
No odd cycles $\Rightarrow$ no conflicts.



Is this a "constructive" argument?

→ Only one component of G is colored.

To color the whole graph, need to pick one starting vtx in every component.

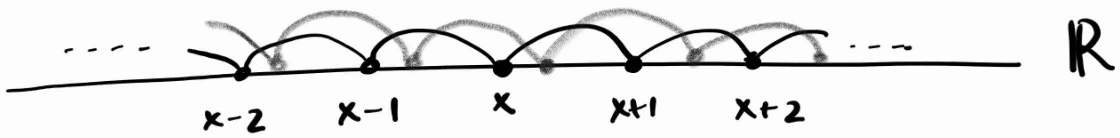
Axiom of Choice: If $\mathcal{F}$ is a family of non-$\varnothing$ sets, then there is a function c s.t. for all $A \in \mathcal{F}$, $\boxed{c(A) \in A}$

c "picks out" one elt from each $A \in \mathcal{F}$.

Non-constructive!

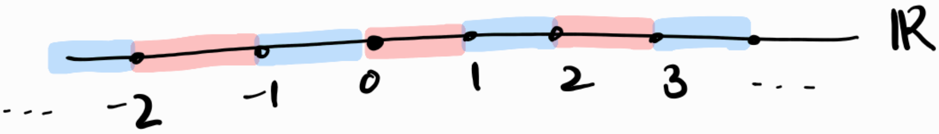
Example 1: $T : \mathbb{R} \to \mathbb{R}$, $T(x) := x+1$.

$G_T :=$ vertex set $\mathbb{R}$, edge set $\{\{x, T(x)\} : x \in \mathbb{R}\}$.



- every component is a bi-infinite path $\Rightarrow$ no cycles $\Rightarrow$ bipartite
- continuum-many components.

A 2-coloring: explicit! ☺

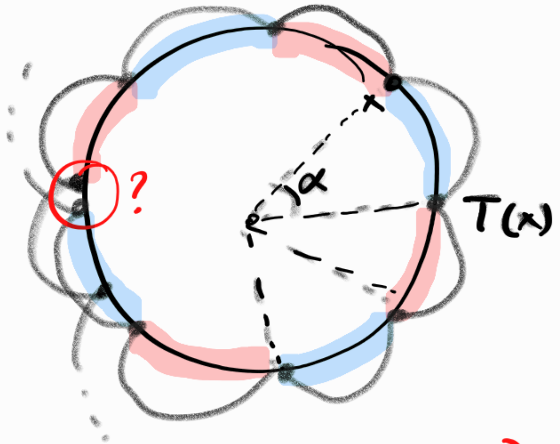


fixed

Example 2: $T : \mathbb{S}^1 \to \mathbb{S}^1$, $T(x) :=$ rotate by angle α

circle

$G_T :=$ vertex set $\mathbb{S}^1$, edge set $\{\{x, T(x)\} : x \in \mathbb{S}^1\}$.



Assume $\frac{\alpha}{\pi} \notin \mathbb{Q}$.

Then every component is a bi-infinite path, continuum-many components (isomorphic to Example 1)

$\Rightarrow$ 2-colorable.

Explicit 2-coloring? **Exercise:** Give an explicit 3-coloring

$\boxed{\text{Claim: No "explicit" 2-coloring exists!}}$ Need to define "explicit"....

Defn: X a topological space.

- σ-algebra on X: A collection of subsets of X closed under complements & ctbl unions (hence also ctbl intersections)
- Borel σ-algebra $\mathcal{B}(X)$: smallest σ-algebra containing all open sets.

A set $A \subseteq X$ is Borel if $A \in \mathcal{B}(X)$.

Ex: $T : \mathbb{R} \to \mathbb{R}$, $T(x) = x+1$

$\mathbb{R} = R \cup B$

Borel sets

$R := \bigcup_{k \in \mathbb{Z}} [2k, 2k+1)$

$B := \bigcup_{k \in \mathbb{Z}} [2k+1, 2k+2)$

On the other hand:

For G_T where $T : S^1 \to S^1$ is rotation by $\alpha \notin \pi\mathbb{Q}$, there is no Borel 2-coloring. (We'll prove this soon)

Another example:

Banach–Tarski Paradox: It is possible to partition a unit ball in $\mathbb{R}^3$ into finitely many pieces, move them by rigid motions, and assemble two disjoint unit balls.

Key: the pieces are $\boxed{\text{not measurable}}$ (so the volume need not be preserved)

↳ Non-Constructive!

(Also uses AC...)

Banach: No such paradox in $\mathbb{R}$ or $\mathbb{R}^2$:
The Lebesgue measure on $\mathbb{R}^d$ for $d \leq 2$ can be extended to an isometry-invariant finitely additive measure defined on all subsets of $\mathbb{R}^d$.

Problem: Are (say)  and  of same area equidecomposable?
(Tarski's circle squaring problem)

Thm (Laczkovich '90): YES! (only using translations)

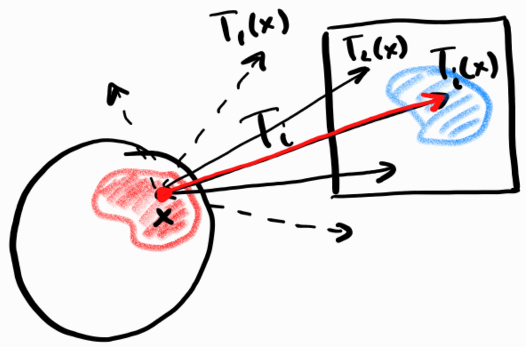
↑ Non-constructive proof, with non-measurable pieces ....

But why?

Thm (Marks–Unger '16): YES with Borel pieces.

Combinatorics connection:

Pick a finite set $T_1, \dots, T_n$ of translations, try to find the pieces that work with them.



Define a bipartite graph G

vertices = circle $\cup$ square

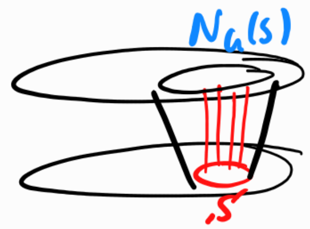
each $x \in$ circle adjacent to all $T_i(x) \in$ square

circle $= A_1 \cup A_2 \cup \dots \cup A_n$

Observation: A suitabe partition exists $\iff$ G has a $\boxed{\text{perfect matching}}$ (i.e., a set of edges s.t. every vtx is incident to exactly one of them)

Graph theory result: A locally finite bipartite graph G has a PM

$\iff$ for all finite $S \subseteq V(G)$, $\underbrace{|\underbrace{N_G(S)}_{\text{nbhd of } S}| \geq |S|}_{\text{Hall's condition}}$.



Given $S \subseteq$ circle, finite, need a lower bound on $|N_G(S)| = |\{T_i x : x \in S\} \cap \text{square}|$ $\rightsquigarrow$ Laczkovich's proof

But: The extension of Hall's thm to loc. finite graphs uses AC ☹

$\Rightarrow$ Need other tools for Borel perfect matchings!