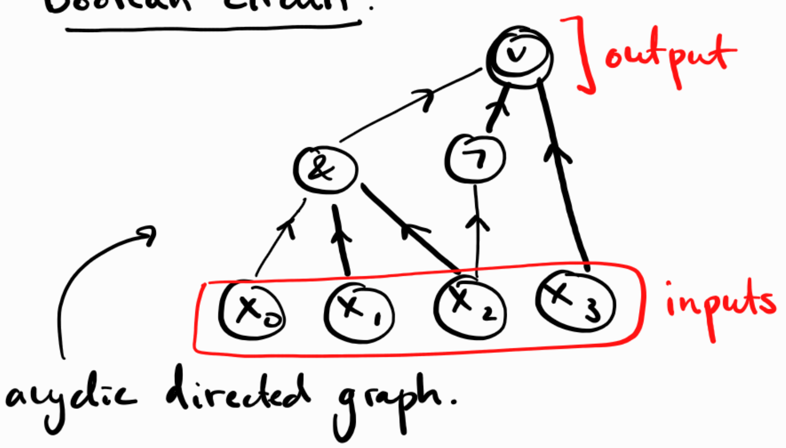
Math 223D, Lecture 2 | Introduction to Borel sets

"Computational perspective": Computation with ctbl resources.

Boolean circuit:



circuit complexity of $f:\{0,1\}^n \to \{0,1\}$

= min # of gates in a circuit that computes f.

P/poly: functions of polynomial circuit complexity.

Idea: Replace "polynomial" w/ "ctbl"

Defn: A Boolean pseudo-circuit: a directed graph C with

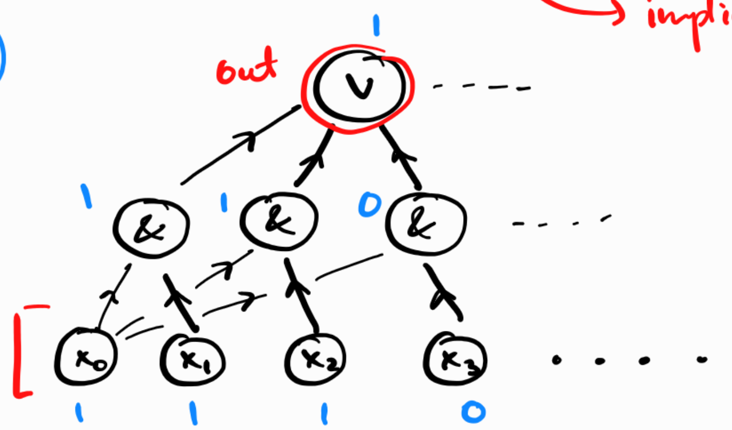
- a subset of $V(C)$ called input nodes
- a vtx called output node
- a label for every non-input node: and-, or-, or not-gate.

The indegree of a not-gate is 1.

A circuit is a pseudo-circuit that is well-founded: no ∞ sequence of nodes $x_0 \leftarrow x_1 \leftarrow x_2 \leftarrow \cdots$

$\rightarrow$ implies C is acyclic

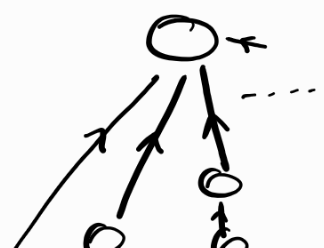
Ex:

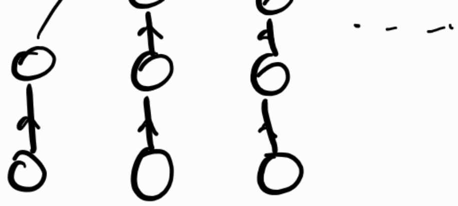


$\rightsquigarrow f:\{0,1\}^{\mathbb{N}} \to \{0,1\}$

$f(x) = 1 \iff x_0 = 1$ and $x_i = 1$ for some $i > 0$.

Note: There may be arbitrarily long descending paths:





Let C be a pseudo-circuit w/ input nodes $I \subseteq V(C)$.
An evaluation of C on input $x = (x_i)_{i \in I} \in \{0,1\}^I$ is a map

$\mathrm{eval}: V(C) \to \{0,1\}$ s.t.

- $\mathrm{eval}(i) = x_i$ for all $i \in I$
- for every gate v, $\mathrm{eval}(v)$ is the result of applying the corresp. Boolean op. to the values of eval on the in-nbrs of v.

[**Proposition:** If C is a circuit w/ inputs I, then C admits a unique evaluation on every input $x \in \{0,1\}^I$]

↳ Define $C(x) := \mathrm{eval}(o)$, where o is the output node.

C computes a function $f: \{0,1\}^I \to \{0,1\}$ if $C(x) = f(x)\ \forall x$.

[**Defn:** Let I be a ctbl set. A set $A \subseteq \{0,1\}^I$ is **Borel** if its charact. function is computed by a ctbl circuit.]

$\mathcal{B}(\{0,1\}^I) := \{$ Borel sets $A \subseteq \{0,1\}^I\}$.

$2 = \{0,1\}$: 2^I

$2^{\mathbb{N}} =$ Cantor space $= \mathcal{C}$.

$(\mathcal{C}, \mathcal{B}(\mathcal{C}))$ ← Main object!

Examples:

- $\{x \in \mathcal{C} : x \text{ has finitely many } 1\text{s}\}$ ← Borel

$\iff \forall i \in \mathbb{N}\ \exists j \geq i\ (x_j = 1)$

$$\iff \bigwedge_{i\in\mathbb{N}} \bigvee_{j\geq i} x_j$$

Borel

- Fix $\delta\in[0,1]$. $\{x\in\mathcal{C} : \lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} x_i = \delta\} =: D_\delta$

$$x\in D_\delta \iff \underbrace{\forall k\geq 1}_{\wedge}\ \underbrace{\exists n_0\in\mathbb{N}}_{\vee}\ \underbrace{\forall n\geq n_0}_{\wedge} \Big(\underbrace{\Big|\frac{1}{n}\sum_{i=0}^{n-1} x_i - \delta\Big| < \frac{1}{k}}\Big)$$

Depends only on $x_0,\dots,x_{n-1}$

$\Rightarrow$ can be computed by a circuit w/ finitely many gates (exercise!)

- $D := \bigcup_{\delta\in[0,1]} D_\delta = \{x\in\mathcal{C} : \lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1} x_i \text{ exists}\}$?

$$x\in D \iff \underbrace{\exists\delta\in[0,1],}_{\text{unctbl!}} (x\in D_\delta)$$

But D is Borel: use that $x\in D \iff \left(\frac{1}{n}\sum_{i<n} x_i\right)_{n\geq 1}$ is Cauchy

(exercise)

Prop: There exists non-Borel subsets of $\mathcal{C}$.

Pf: $|\mathcal{P}(\mathcal{C})| = 2^{2^{\aleph_0}}$

$|\mathcal{B}(\mathcal{C})| \leq$? Count circuits!

WLOG may only consider circuits with fixed vertex set V where $V = \{p_i, a_i, r_i, n_i\}_{i\in\mathbb{N}}$ s.t.:

1. the inputs are $\{p_i\}$
2. each $a_i / r_i / n_i$ is an and/or/not gate
3. the output node is a_0.

④ the unique in-nbr of n_i is p_i. (de Morgan)

Such a circuit is determined by its edge set.

List $\{(u,v)\in V\times V : v\notin\{n_i\}_{i\in\mathbb{N}}\}$

Encode C by $x_C=(x_{C,0}, x_{C,1}, \dots)\in\mathcal{C}$

where $x_{C,i}=1 \iff e_i\in E(C)$.

$\Rightarrow$ there are $\le |\mathcal{C}| = 2^{\aleph_0}$ such circuits $\Rightarrow |\mathcal{B}(\mathcal{C})|\le 2^{\aleph_0}$]

Note: Each $x\in\mathcal{C}$ corresp. to a unique pseudo-circuit C_x:

$$E(C_x) := \{e_i : x_i = 1\}\cup\{(p_i,n_i) : i\in\mathbb{N}\}.$$

$$\boxed{A := \{x\in\mathcal{C} : C_x \text{ is well-founded } \& \ C_x(x)=0\}.}$$

Prop: A is not Borel.

Proof: Otherwise there is $x\in\mathcal{C}$ s.t. C_x computes $\mathbb{1}_A$.

$x\in A \iff C_x(x)=1 \iff x\notin A$.