Math 223D, Lecture 31

Coding of countable pseudo-circuits by points in $\mathcal{C} = 2^{\mathbb{N}}$:

$$x \longmapsto C_x$$

$$A := \{ x \in \mathcal{C} : C_x \text{ is well-founded and } C_x(x) = 0 \}$$

Claim: A is not a Borel set.

Pf: If it is, $\mathbb{1}_A$ is computed by C_x for some $x \in \mathcal{C}$. Then

$$x \in A \iff C_x(x) = 0 \iff x \notin A$$

Remarks:

① The set $\{x \in \mathcal{C} : C_x \text{ is well-founded}\}$ is already non-Borel.

② Can code various objects by pts in $\mathcal{C}$

- **Graphs** $= \{$all graphs with vtx set $\mathbb{N}\} \cong \mathcal{C}$

  List all edges of the complete graph $K(\mathbb{N})$ as $e_0, e_1, e_2, \dots$

  $$\mathcal{C} \ni x \longmapsto G_x \text{ with edge set } \{e_i : x_i = 1\}.$$

- **Groups** $= \{$all groups with underlying set $\mathbb{N}\}$

  Need to encode the operation: Identify G with the set $\{(x,y,z) \in \mathbb{N}^3 : x \cdot y = z\} \subseteq \mathbb{N}^3$ ($\cdot$ = group operation)

  So $\underbrace{\text{Groups}}_{} \subseteq 2^{\mathbb{N}^3} \cong \mathcal{C}$.

  Is it Borel? Yes:

  A set $S \subseteq \mathbb{N}^3$ codes a group

  $$\iff \forall x \in \mathbb{N}\ \forall y \in \mathbb{N}\ \exists! z \in \mathbb{N}\ ((x,y,z) \in S)$$

  & etc. (exercise!)

- $C(\mathbb{R}) = \{$continuous functions $f : \mathbb{R} \to \mathbb{R}\}$

  $(f(q_0), f(q_1), \dots) \in \mathbb{R}^{\mathbb{N}}$ where

$\text{code}(f) := (f(q_0), f(q_1), \dots) \in \mathbb{R}^{\omega}$,

$q_0, q_1, \dots$ is an enumeration of $\mathbb{Q}$.

Exercise: The set of all codes is Borel.

Borel Functions

Defn: $f : \mathcal{C} \to \mathcal{C}$ is Borel if there exist ctbl circuits $(C_i)_{i \in \mathbb{N}}$ s.t. for all $x \in \mathcal{C}$, $C_i(x)$ is the i^{th} bit of $f(x)$.

Prop: $f : \mathcal{C} \to \mathcal{C}$ is Borel $\iff$ $\forall$ Borel $A \subseteq \mathcal{C}$, $f^{-1}(A)$ is Borel.

(exercise)

IMPORTANT! For Borel $f : \mathcal{C} \to \mathcal{C}$ and Borel $A \subseteq \mathcal{C}$, the image $f(A)$ need not be Borel!

Example: Let $A := \{x \in \mathcal{C} : C_x \text{ is well-founded } \& \ C_x(x) = 0\}$

non-Borel set from earlier

Then A^c is the image of a Borel set under a Borel function.

(we'll get back to this)

However:

Thm (Luzin-Suslin): If $A \subseteq \mathcal{C}$ is Borel and $f : A \to \mathcal{C}$ is an injective Borel function, then $f(A)$ is Borel.

$\text{graph}(f) := \{(x,y) \in \mathcal{C} \times \mathcal{C} : f(x) = y\}$.

often easier to check

Thm: $f : \mathcal{C} \to \mathcal{C}$ is Borel $\iff$ $\text{graph}(f)$ is a Borel subset of $\mathcal{C} \times \mathcal{C}$.

Pf: ($\Rightarrow$) Let $(C_i)_{i \in \mathbb{N}}$ compute f.

Give a ... for $(x,y) \in \text{graph}(f)$ as follows:

Given $x, y \in \mathcal{C}$, decide if $(x,y) \in \operatorname{graph}(f)$ as follows:
Compute $(C_i(x))_{i \in \mathbb{N}}$ and compare each $C_i(x)$ with y_i.

$\Leftarrow$ Take a Borel set $A \subseteq \mathcal{C}$ and consider the map

$$\begin{aligned} (\mathcal{C} \times A) \cap \operatorname{graph}(f) &\longrightarrow \mathcal{C} \\ (x,y) &\longmapsto x \end{aligned}$$

This is an injective Borel map whose image is $f^{-1}(A)$.

Topological Perspective

$2 = \{0,1\}$ discrete topology

$\mathcal{C} = 2^{\mathbb{N}}$ product topology: smallest topology s.t. projections $\mathrm{proj}_i : 2^{\mathbb{N}} \to 2$ are continuous.

Normalized Hamming metric: summable $\Rightarrow d(x,y) \le 1$.

$$d(x,y) := \sum_{i=0}^{\infty} \frac{1}{2^{i+1}} |x_i - y_i|$$

For $x \in \mathcal{C}$ and $D \subseteq \mathbb{N}$, let $N_D(x) := \{ y \in \mathcal{C} : y \text{ agrees w/ } x \text{ on } D \}$

[**Lemma:** $\forall x \in \mathcal{C}$, $\{ N_D(x) : D \subset \mathbb{N} \text{ finite} \}$ is a nbhd basis at x.]

Pf: ⓐ For each $\varepsilon > 0$, $\exists$ finite $D \subset \mathbb{N}$ s.t.

$$N_D(x) \subseteq B_\varepsilon(x).$$

Why? Take n so large that $\varepsilon > \frac{1}{2^n}$, $D := \{0, 1, \dots, n-1\}$.

$$y \in N_D(x) \Rightarrow d(x,y) \le \sum_{i \ge n} \frac{1}{2^{i+1}} |x_i - y_i| \le \frac{1}{2^n} < \varepsilon.$$

ⓑ $\forall$ finite $D \subset \mathbb{N}$, $\exists\, \varepsilon > 0$ s.t. $B_\varepsilon(x) \subseteq N_D(x)$ exercise!

Say $U \subseteq \mathcal{C}$ is determined by $D \subseteq \mathbb{N}$ if for all $x \in \mathcal{C}$, membership of x in U only depends on $(x_i)_{i \in D}$.

$\mathcal{U}$ is finitely determined if it is determined by a finite set.

($\Longleftrightarrow$ $1_{\mathcal{U}}$ is computed by a circuit with finitely many gates)

[**Lemma** Every finitely determined set is clopen.] $\rightsquigarrow$ closed & open

Pf: $\mathcal{U} \subseteq \mathcal{C}$ determined by $D \subset \mathbb{N}$ finite.

Open: $\mathcal{U} \supseteq N_D(x)$ for each $x \in \mathcal{U}$.

Closed: $\mathcal{U}^c$ is also fin. det.

[**Thm:** A set $\mathcal{U} \subseteq \mathcal{C}$ is clopen $\Longleftrightarrow$ it is fin. det.]

Pf: exercise!

[**Thm:** The family of fin det sets forms a basis for the topology on $\mathcal{C}$.]

Pf: Take $\mathcal{U} \subseteq \mathcal{C}$ open. Then $\forall x \in \mathcal{U}$, there is finite $D(x) \subset \mathbb{N}$ s.t. $N_{D(x)}(x) \subseteq \mathcal{U}$. Hence $\mathcal{U} = \bigcup_{x \in \mathcal{U}} N_{D(x)}(x)$.

[**Lemma:** The metric d on $\mathcal{C}$ is complete (every Cauchy seq. has a limit)] $\rightsquigarrow$ exercise.

Summary:

1. Top. on $\mathcal{C}$ is generated by a complete metric.
2. $\mathcal{C}$ is second-ctbl: there is a countable basis for the topology
3. $\mathcal{C}$ is zero-dimensional: a basis of clopen sets
4. $\mathcal{C}$ is compact (exercise)

[**Prop:** $\mathcal{B}(\mathcal{C})$ = smallest σ-algebra on $\mathcal{C}$ containing all open sets.]

Pf: exercise

$\longrightarrow$ Can define Borel sets in an arbitrary top. space.