Math 223D, Lecture 41

Topology of $\mathcal{C} = 2^{\mathbb{N}}$

Lemma Every finitely determined set is clopen. ↳ closed & open

Pf: $U \subseteq \mathcal{C}$ determined by $D \subset \mathbb{N}$ finite.

Open: $U \supseteq N_D(x)$ for each $x \in U$.

Closed: U^c is also fin. det.

Thm: A set $U \subseteq \mathcal{C}$ is clopen $\Leftrightarrow$ it is fin. det.

Pf: exercise!

Thm: The family of fin det sets forms a basis for the topology on $\mathcal{C}$.

Pf: Take $U \subseteq \mathcal{C}$ open. Then $\forall x \in U$, there is finite $D(x) \subset \mathbb{N}$ s.t. $N_{D(x)}(x) \subseteq U$. Hence $U = \bigcup_{x \in U} N_{D(x)}(x)$.

Lemma: The metric d on $\mathcal{C}$ is complete (every Cauchy seq. has a limit) ⇝ exercise.

Summary:

① Top. on $\mathcal{C}$ is generated by a complete metric.

② $\mathcal{C}$ is second-ctbl: there is a countable basis for the topology

③ $\mathcal{C}$ is zero-dimensional: a basis of clopen sets

④ $\mathcal{C}$ is compact (exercise)

Prop: $\mathcal{B}(\mathcal{C})$ = smallest σ-algebra on $\mathcal{C}$ containing all open sets.

Pf: exercise

→ Can define Borel sets in an arbitrary top. space.

Defn: A top. space X is **Polish** if:

1. the topology is generated by a complete metric
2. X is second-ctbl.

Examples: $\mathcal{C}$, $\mathbb{R}$, $\mathbb{R}^n$, $[0,1]$, $(0,1)$, $\mathbb{R}\setminus\mathbb{Q}$, ...

exercise!

Lemma: X Polish $\Rightarrow$ $\exists$ a compatible complete metric bounded by 1.

Pf: $\tilde{d}(x,y) := \min\{d(x,y), 1\}$.

- Ctbl sums (i.e., disjoint unions) and products of Polish spaces are Polish.

Thm (Alexandrov): X a Polish space, $A \subseteq X$.
A is Polish (in the subspace top) $\Longleftrightarrow$ A is G_δ.

ctbl $\bigcap$ of open sets

F_σ = ctbl $\bigcup$ of closed = complement of G_δ.

Why Polish spaces?

Thm X a Polish space $\Rightarrow$ there is a map code: $X \to \mathcal{C}$ s.t. $B \subseteq X$ is Borel in X iff $\{\text{code}(x) : x \in B\}$ is Borel in $\mathcal{C}$.

What is code?

Let $(U_n)_{n\in\mathbb{N}}$ be a basis for the top. on X.

$\Big\{ 1$ if $x \in U_n$

Define $\text{code}(x) := (\text{code}_n(x))_{n\in\mathbb{N}}$ where $\text{code}_n(x) = \{ 0$ if $x \notin U_n$.

Now, let $\Gamma := \text{graph}(\text{code}) \subseteq X \times \mathcal{C}$.

$(x,y) \in \Gamma \iff y = \text{code}(x) \iff$

$$\underbrace{\forall n \in \mathbb{N}}_{\text{ctbl } \cap} \left(\left(\underbrace{y_n = 0}_{\substack{\text{Borel} \\ \text{(clopen)}}} \ \overset{\cap}{\&} \ \underbrace{x \notin U_n}_{\text{closed}} \right) \overset{\cup}{\vee} \left(\underbrace{y_n = 1}_{\text{clopen}} \ \overset{\cap}{\&} \ \underbrace{x \in U_n}_{\text{closed}} \right) \right)$$

So Γ is a Borel set, hence code is a Borel function.

Since code is injective (exercise),

$$B \subseteq \mathcal{C} \text{ is Borel} \iff \text{code}(B) \text{ is Borel}$$

by Luzin–Suslin. $\boxtimes$

More is true:

Thm (Borel isomorphism theorem):
Let X be a Polish space, $A \subseteq X$ Borel, uncountable.
Then $(A, \mathcal{B}(A)) \cong (\mathcal{C}, \mathcal{B}(\mathcal{C}))$

Defn: A *standard Borel space* is a structure isomorphic to $(X, \mathcal{B}(X))$ for a Polish space X.

→ either $(X, \mathcal{P}(X))$ for ctbl X

or $\cong (\mathcal{C}, \mathcal{B}(\mathcal{C}))$.

Changing the topology

X a standard Borel space. A topology τ on X is *compatible* if $\mathcal{B}(X) =$ Borel σ-algebra of (X, τ).

Notation: $\langle \mathcal{U} \rangle =$ top. generated by $\mathcal{U}$.

Thm: If $(\tau_n)_{n\in\mathbb{N}}$ are compatible Polish topologies on st. Borel X, then $\tau_\infty := \langle \bigcup_{n\in\mathbb{N}} \tau_n \rangle$ is also a compatible Polish top.

Pf: τ_∞ is clearly compatible, second-ctbl.

Fix complete metrics d_n bounded by 1 s.t. d_n generates τ_n.

$$d_\infty(x,y) := \sum_{n\in\mathbb{N}} \frac{1}{2^{n+1}} d_n(x,y)$$ is a complete metric that generates τ_∞ (exercise) ☒

Thm (MAKING BOREL SETS CLOPEN): X standar Borel. $\forall A \in \mathcal{B}(X)$, there is a compatible Polish τ on X s.t. A is τ-clopen.

Pf: Let $\mathcal{A} := \{ A \in \mathcal{B}(X) :$ there is a compatible Polish τ on X s.t. A is τ-clopen $\}$.

Fix a compatible Polish top. τ on X. Need to show:

① Every τ-open set is in $\mathcal{A}$

② $\mathcal{A}$ is a σ-algebra

$\Rightarrow \mathcal{A} \supseteq \mathcal{B}(X)$ ☺

①: $U \subseteq X$ τ-open $\Rightarrow$ $\underset{\text{open}}{U}$ Polish, $\underset{\text{closed}}{X\setminus U}$ Polish

$X \cong \underbrace{U \sqcup (X\setminus U)}$ Polish, U and $X\setminus U$ both open. ✓

② $A \in \mathcal{A} \Rightarrow A^c \in \mathcal{A}$ clear

$(A_n \in \mathcal{A})_{n\in\mathbb{N}} \overset{?}{\Rightarrow} \bigcup A_n \in \mathcal{A}$

Let τ_n witness $A_n \in \mathcal{A}$

$\tau_\infty := \langle \bigcup_{n\in\mathbb{N}} \tau_n \rangle$ compatible, Polish

In τ_∞, each A_n is clopen $\Rightarrow A$ is τ_∞-open

$\Rightarrow A \in \mathcal{A}$ by ①. ☒