MATH 223D, Lecture 5

Changing the topology

X a standard Borel space. A topology τ on X is **compatible** if $\mathcal{B}(X) =$ Borel σ-algebra of (X,τ).

Notation: $\langle \mathcal{U} \rangle =$ top. generated by $\mathcal{U}$.

Thm: If $(\tau_n)_{n\in\mathbb{N}}$ are compatible Polish topologies on st. Borel X, then $\tau_\infty := \left\langle \bigcup_{n\in\mathbb{N}} \tau_n \right\rangle$ is also a compatible Polish top.

Pf: τ_∞ is clearly compatible, second-ctbl.

Fix complete metrics d_n bounded by 1 s.t. d_n generates τ_n.

$$d_\infty(x,y) := \sum_{n\in\mathbb{N}} \frac{1}{2^{n+1}} d_n(x,y)$$ is a complete metric that generates τ_∞ (exercise)

Thm (MAKING BOREL SETS CLOPEN): X standar Borel. $\forall A \in \mathcal{B}(X)$, there is a compatible Polish τ on X s.t. A is τ-clopen.

Pf: Let $\mathcal{A} := \{ A \in \mathcal{B}(X) : \quad \}$.

Fix a compatible Polish top. τ on X. Need to show:

① Every τ-open set is in $\mathcal{A}$

② $\mathcal{A}$ is a σ-algebra

$\Big\} \Rightarrow \mathcal{A} \supseteq \mathcal{B}(X)$ ☺

①: $U \subseteq X$ τ-open $\Rightarrow$ $\underset{\text{open}}{U}$ Polish, $\underset{\text{closed}}{X\setminus U}$ Polish

$X \cong \underbrace{U \sqcup (X\setminus U)}$ Polish, U and $X\setminus U$ both open. ✓

② $A \in \mathcal{A} \Rightarrow A^c \in \mathcal{A}$ clear

$(A_n \in \mathcal{A})_{n\in\mathbb{N}} \Rightarrow \bigcup A_n \in \mathcal{A}$

Let τ_n witness $A_n \in \mathcal{A}$

$$\tau_\infty := \left\langle \bigcup_{n\in\mathbb{N}} \tau_n \right\rangle \text{ compatible, Polish}$$

In τ_∞, each A_n is clopen $\Rightarrow A$ is τ_∞-open

$\Rightarrow A \in \mathcal{A}$ by ①. ■

Corollary: X st. Borel, (Y, τ_Y) Polish, $f: X \to Y$ a Borel function. Then $\exists$ a compatible Polish top. τ_X on X s.t. $f: (X, \tau_X) \to (Y, \tau_Y)$ is $\boxed{\text{continuous}}$.

Pf: $\mathcal{U}$ = ctbl basis for τ_Y. Enough to find τ_X s.t.

$\forall U \in \mathcal{U}$, $f^{-1}(U)$ is τ_X-open.

$f^{-1}(U)$ is Borel $\Rightarrow \exists\, \tau_U$ comp. Polish top. on X s.t. U is τ_U-clopen

$\tau_X := \left\langle \bigcup_{U\in\mathcal{U}} \tau_U \right\rangle$ is as desired. ■

Analytic & co-analytic sets

(analogous to NP)

Defn: X a standard Borel space. $A \subseteq X$ is *analytic* if there is a standard Borel space Y, Borel set $B \subseteq X \times Y$ s.t.

$$A = \text{proj}_X(B) := \{x \in X : \underline{\exists y \in Y}\, ((x,y) \in B)\}$$

Equivalently:

$A \subseteq X$ is analytic if $\boxed{A = f(Y)}$ for a st. Borel Y, a Borel function $f: Y \to X$.

$A \subseteq X$ is <u>co-analytic</u> if for some Borel $B \subseteq X \times Y$,

$$A = \{ x \in X : \forall y \in Y \, ((x,y) \in B) \}.$$

Equivalently, A^c is analytic.

Example: <u>Graphs</u> $\cong \mathcal{C}$ ← graphs with $V = \mathbb{N}$.

$\underline{\text{Clique}}_\infty := \{ G \in \underline{\text{Graphs}} : G \text{ has an infinite clique} \}$ is analytic.

$G \in \underline{\text{Clique}}_\infty \iff \exists$ infinite set $S \subseteq \mathbb{N} \, (G[S] \text{ is complete})$

$$\iff \exists x \in \mathcal{C} \, \Big(\underbrace{x \text{ has } \infty \text{ many 1s}}_{\text{Borel}} \wedge \underbrace{\forall i < j \in \mathbb{N}}_{\text{ctbl } \cap} \underbrace{(x_i = x_j = 1 \to \overbrace{G_{ij} = 1}^{\{i,j\} \in E(G)})}_{\text{finitely determined} \Rightarrow \text{Borel}} \Big)$$

Another example:

$\underline{\text{Dir}} = \{ \text{directed graphs with vtx set } \mathbb{N} \} \cong \mathcal{C}$.

$\underline{\text{WF}} := \{ D \in \underline{\text{Dir}} : D \text{ is } \underline{\text{well-founded}} \}$

well-founded ↳ no $\bullet \leftarrow \bullet \leftarrow \bullet \leftarrow \bullet \leftarrow \cdots$

Then <u>WF</u> is <u>co-analytic</u>.

$$D \in \underline{\text{WF}} \iff \neg \exists x \in \mathcal{N} = \mathbb{N}^{\mathbb{N}} \, \underbrace{\forall i \in \mathbb{N} \, ((x_{i+1}, x_i) \in E(D))}_{\text{Borel}}.$$

$\mathcal{N}$ ↑ the Baire space

Examples from analysis:

$C([0,1]) := \{ \text{continuous functions } f : [0,1] \to \mathbb{R} \} \cong \mathcal{C}$

(record the values of f on the rational pts)

← exercise

The following [illegible] $C([0,1])$ Polish:

The following metric makes $C([0,1])$ Polish:

$$\mathrm{dist}(f,g) := \sup |f(x)-g(x)| \quad \text{(uniform convergence)}$$

$$PC_0 := \underbrace{\{ (f_n) \in C([0,1])^{\mathbb{N}} : (f_n) \text{ converges ptwise to } 0 \}.}_{\text{co-analytic:}}$$

$$(f_n) \in PC_0 \iff \underset{=}{\forall} x \in [0,1] \big(\underbrace{f_n(x) \longrightarrow 0}_{\text{Borel}} \big)$$

$$\mathrm{Diff} := \{ f \in C([0,1]) : f \text{ is differentiable} \} \rightsquigarrow \text{co-analytic}$$

$$\forall x \in [0,1] \; \big(\underbrace{f \text{ is differentiable at } x}_{\text{Borel (exercise!)}} \big)$$

Thm (Suslin): Not all analytic / co-analytic sets are Borel.

(set-theorist's $P \neq NP$)

Pf idea: Recall coding of pseudo-circuits by pts in $\mathcal{C}$

$\mathcal{C} \ni x \longmapsto C_x$, fixed ctbl vtx set V

$$A := \{ x \in \mathcal{C} : C_x \text{ is well-founded } \& \; C_x(x) = 0 \}.$$

NOT BOREL

$$x \in A \iff \underbrace{C_x \text{ is well-founded}}_{\text{co-analytic}} \;\&\; \underset{=}{\forall} f : V \to \{0,1\}$$

a pt in $2^V \cong 2^{\mathbb{N}}$

(f is an evaluation of C_x on x and $f(\text{output}) = 0$)

Borel

So A is co-analytic. $\square$

Actually, $\underline{\text{Clique}_\infty}$, $\underline{WF}$, $\underline{PC_0}$, $\underline{\mathrm{Diff}}$ are not Borel!