MATH 223D, Lecture 6

Analytic sets: Images of Borel functions / $\exists y \in Y (\dots)$

Co-analytic: Complements of analytic / $\forall y \in Y (\dots)$

Suslin: Not all analytic/co-analytic sets are Borel.

Defn: Let X, Y be standard Borel spaces, $A \subseteq X$, $B \subseteq Y$.
A **Borel reduction** from A to B is a Borel map $f: X \to Y$ s.t. $A = f^{-1}(B)$, i.e., $(x \in A \iff f(x) \in B)$.

$A \subseteq X$ is **complete analytic** if A is analytic & every analytic set is Borel reducible to A. Same for **complete coanalytic**.

Facts: ① A complete analytic $\iff$ A^c is complete co-analytic

② complete analytic/co-analytic set can't be Borel
(because a non-Borel set is Borel reducible to it)

Thm: The set $\underline{WF}$ of well-founded digraphs is complete co-analytic

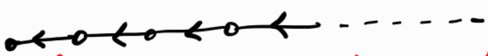
Proof: Given st. Borel X, Y and a Borel map

$$f: X \to Y$$

want a Borel reduction from $f(X)$ ← analytic to $\underline{WF}^c$.

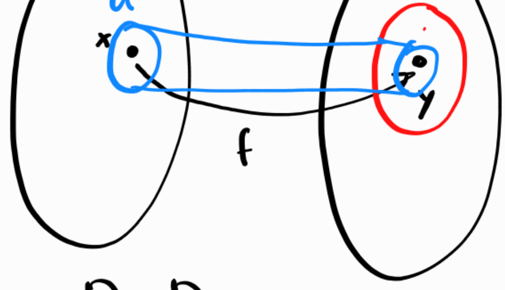
May assume: X, Y Polish with complete metrics d_X, d_Y, f is $\boxed{\text{continuous}}$

To each $y \in Y$, assign (in a Borel way) a ctbl digraph D_y so that $y \in f(X) \iff D_y$ is not well-founded



← "encode" a preimage of y

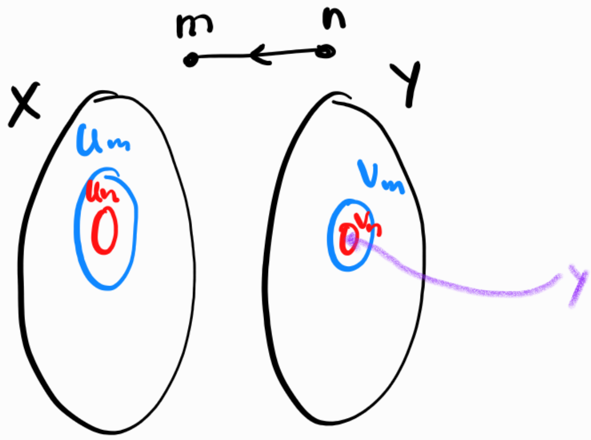
Let $\mathcal{U}$, $\mathcal{V}$ be ctbl bases for X, Y.



Let (U_n, V_n) be an enumeration of all pairs $U_n \in \mathcal{U}$, $V_n \in \mathcal{V}$ s.t. $f(U_n) \subseteq V_n$ (independent of y)

Define $D \in \underline{\mathrm{Dir}}$:

In D, have an edge (n, m) if and only if:



① $cl(U_n) \subseteq U_m$

closure

$cl(V_n) \subseteq V_m$

② $\mathrm{diam}(U_n) \leq \frac{1}{2}\mathrm{diam}(U_m)$

$\mathrm{diam}(V_n) \leq \frac{1}{2}\mathrm{diam}(V_m)$

Again, this is indep. of y.

Finally: D_y has edge set $\{(n,m) \in E(D) : y \in V_n\}$.

$(n,m) \in D_y \iff \underbrace{(n,m) \in D}_{\text{constant}}$ & $\underbrace{y \in V_n}_{\text{open}}$, so the map

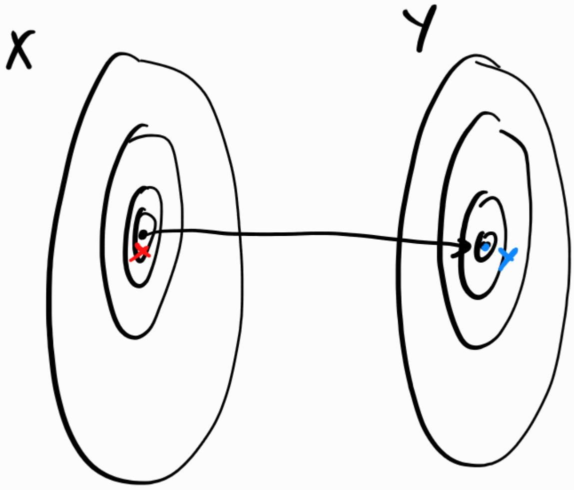
$y \mapsto D_y$ is Borel.

If D_y is not WF: $n_0 \leftarrow n_1 \leftarrow n_2 \leftarrow \cdots$:

$U_{n_0} \supseteq \overline{U}_{n_1} \supseteq \overline{U}_{n_2} \supseteq \cdots$

diameters go to 0 $\Rightarrow$ there is a unique pt x in $\bigcap_{i \in \mathbb{N}} U_{n_i}$.

completeness of the metric



$V_{n_0} \supseteq \overline{U}_{n_1} \supseteq \cdots \ni \underline{\underline{y}}$

$f(x) \in f(U_{n_i}) \subseteq V_{n_i}$ $\forall i$,

so $f(x) = y$, i.e., $y \in f(X)$.

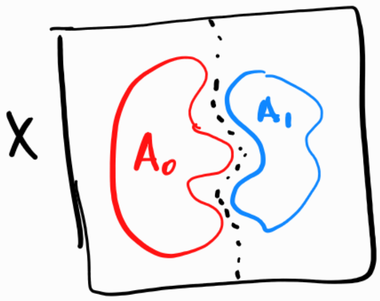
Conversely, if $y \in f(X)$, then D_y is not WF (exercise.

A is co-analytic and $\underline{WF}$ is Borel reducible to A

$\Rightarrow$ A is complete co-analytic

Can show $\underline{\text{Clique}_\infty}$, $\underline{PC_0}$, $\underline{\text{Diff}}$ are complete.

Thm (Analytic Separation): X a standard Borel space, $A_0, A_1 \subseteq X$ disjoint analytic sets. Then there exists a partition $X = B_0 \cup B_1$ into Borel sets such that $A_0 \subseteq B_0$, $A_1 \subseteq B_1$.



Corollary: $A \subseteq X$ is both analytic and co-analytic $\Rightarrow$ A is Borel.

Proof: There exist Borel functions $D_0, D_1 : X \to \underline{\text{Dir}}$ such that $A_i = D_i^{-1}(\underline{WF}^c)$. In this language:

For all $x \in X$, at least one of $D_0(x)$, $D_1(x)$ is well-founded

Want: A Borel map $F : X \to \{0,1\}$ s.t. if $F(x) = i$, then $D_i(x)$ is well-founded.

Idea: A recursive definition:

For $u_0, u_1 \in \mathbb{N}$ (the vtx set), define $F(x, u_0, u_1)$ so that if $F(x, u_0, u_1) = i$, then $D_i(x)$ has no infinite descending chain starting from u_i : $u_i \leftarrow \circ \leftarrow \circ \leftarrow \cdots$ ✗

Definition:

$F(x, u_0, u_1) = 0 :\Longleftrightarrow$

$\forall \ldots \xrightarrow{D_0(x)} \ldots \exists \ldots \mathbb{N} \ldots \xrightarrow{D_1(x)} \ldots$ s.t.

$$\forall v_0 \in \mathbb{N} \text{ s.t. } v_0 \longrightarrow u_0 \quad \exists v_1 \in \mathbb{N} \text{ s.t. } v_1 \longrightarrow u_1 \text{ s.t.}$$

$$\Big(F(x, v_0, v_1) = 0 \Big).$$

TBC.