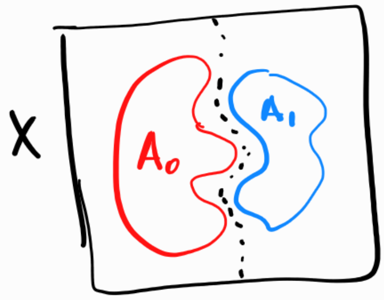
Math 223D, Lecture 7

Thm (Analytic Separation): X a standard Borel space, $A_0, A_1 \subseteq X$ disjoint analytic sets. Then there exists a partition $X = B_0 \sqcup B_1$ into Borel sets such that $A_0 \subseteq B_0$, $A_1 \subseteq B_1$.



Corollary: $A \subseteq X$ is both analytic and co-analytic $\Rightarrow$ A is Borel.

Proof: There exist Borel functions $D_0, D_1 : X \to \mathrm{Dir}$ such that $A_i = D_i^{-1}(\mathrm{WF}^c)$. In this language:

For all $x \in X$, at least one of $D_0(x)$, $D_1(x)$ is well-founded.

Want: A Borel map $F : X \to \{0,1\}$ s.t. if $F(x) = i$, then $D_i(x)$ is well-founded.

Idea: A recursive definition:

For $u_0, u_1 \in \mathbb{N}$ (the vtx set), define $F(x, u_0, u_1)$ so that if $F(x, u_0, u_1) = i$, then $D_i(x)$ has no infinite descending chain starting from u_i : $u_i \leftarrow \circ \leftarrow \circ \leftarrow \cdots$ ✗

Definition:

$$\boxed{\begin{gathered} F(x, u_0, u_1) = 0 :\Longleftrightarrow \\ \forall v_0 \in \mathbb{N} \text{ s.t. } v_0 \xrightarrow{D_0(x)} u_0 \;\; \exists v_1 \in \mathbb{N} \text{ s.t. } v_1 \xrightarrow{D_1(x)} u_1 \text{ s.t.} \\ \big(F(x, v_0, v_1) = 0\big). \end{gathered}}$$

① This is well-defined, i.e., for each $x \in X$, the recursion is well-founded:

To compute $F(x,u_0,u_1)$, need to know $F(x,v_0,v_1)$ for all in-nbrs v_i of u_i in $D_i(x)$

If this is not well-founded, get an ∞ sequence

$(u_{00},u_{01}), (u_{10},u_{11}), (u_{20},u_{21}), \dots$ s.t.

$u_{00} \leftarrow u_{10} \leftarrow u_{20} \leftarrow \cdots$ in $D_0(x)$ and

$u_{01} \leftarrow u_{11} \leftarrow u_{21} \leftarrow \cdots$ in $D_1(x)$

$\Rightarrow$ Neither $D_0(x)$ nor $D_1(x)$ is well-founded $\lightning$

② If $F(x,u_0,u_1)=i$, then in $D_i(x)$ there is no ∞ desc. chain starting with u_i (as desired):

Say $F(x,u_0,u_1)=0$ but $u_0 = u_{00} \leftarrow u_{10} \leftarrow u_{20} \leftarrow \cdots$ is a chain in $D_0(x)$. Set $u_{01} := u_1$.

Now, $F(x,u_0,u_1)=0 \Rightarrow$

$$\Rightarrow \underbrace{\forall v_0 \xrightarrow{D_0(x)} u_0}_{\text{take } v_0 = u_{10}} \; \exists v_1 \xrightarrow{D_1(x)} u_1 \; \big(F(x,v_0,v_1)=0\big)$$

$$\Rightarrow \exists u_{11} \xrightarrow{D_1(x)} u_{01} \text{ s.t. } \boxed{F(x,u_{10},u_{11})=0}$$

By repeating this argument, get a chain

$u_{01} \leftarrow u_{11} \leftarrow u_{21} \leftarrow \cdots$ in $D_1(x)$ $\lightning$

If $F(x,u_0,u_1)=1$, the proof is similar (exercise).

③ F is a Borel function, i.e., it is computed by a ctbl circuit.

First idea: Compute $D_0(x)$, $D_1(x)$.

Then have one node for each pair $(u_0,u_1) \in \mathbb{N}^2$ that computes $F(x,u_0,u_1)$.

This gives a ctbl pseudo-circuit, but it isn't well-founded!

Each (u_0, u_1) depends on all other (v_0, v_1), since depending on x, there may exist edges $v_0 \xrightarrow{D_0(x)} u_0$, $v_1 \xrightarrow{D_1(x)} u_1$

Idea: Use topology to keep track of the edges of $D_0(x)$, $D_1(x)$ we are "committed to"

Assume (WLOG): $D_0, D_1 : X \to \underline{\mathrm{Dir}}$ continuous.

Fix a compatible complete metric on X and a ctbl basis $(U_n)_{n \in \mathbb{N}}$.

Since D_i is continuous, for each $x \in X$ and edge $v_i \longrightarrow u_i$ in $D_i(x)$, there is an open nbhd U of x s.t. $\boxed{v_i \longrightarrow u_i \text{ is present in } D_i(y) \text{ for all } y \in U.}$

"U guarantees $v_i \xrightarrow{D_i} u_i$"

We'll have separate gates $F_n(x, u_0, u_1)$ computing $F(x, u_0, u_1)$ on each U_n using the formula:

assume $x \in U_n$

$$F_n(x, u_0, u_1) = 0 :\iff$$

$$\forall v_0 \in \mathbb{N} \text{ s.t. } v_0 \xrightarrow{D_0(x)} u_0 \;\; \exists v_1 \in \mathbb{N} \text{ s.t. } v_1 \xrightarrow{D_1(x)} u_1 \text{ s.t.}$$

$$\left(\begin{array}{l} F_m(x, v_0, v_1) = 0, \text{ where } m \in \mathbb{N} \text{ is (say) smallest} \\ \text{s.t. } x \in U_m,\ U_m \text{ guarantees } v_0 \xrightarrow{D_0} u_0,\ v_1 \xrightarrow{D_1} u_1, \\ \text{and } \overline{U_m} \subseteq U_n,\ \operatorname{diam} U_m \leq \tfrac{1}{2} \operatorname{diam} U_n. \end{array} \right)$$

$(*)$

Assuming $x \in U_n$, we have $F_n(x, u_0, u_1) = F(x, u_0, u_1)$.

But now, the gate indexed by (n, u_0, u_1) only depends on (m, v_0, v_1) where $(*)$ holds!

Well-founded: Otherwise, have a sequence ...

where $U_{n_0} \supseteq \overline{U}_{n_1} \supseteq \overline{U}_{n_2} \supseteq \cdots$, diam $\to 0$

$\Rightarrow$ have $x \in \bigcap U_{n_i}$.

U_{n_i} guarantees $u_{i0} \xrightarrow{D_0} u_{(i-1)0}$, $u_{i1} \xrightarrow{D_1} u_{(i-1)1}$

so in $D_0(x)$ have $u_{00} \leftarrow u_{10} \leftarrow u_{20} \leftarrow \cdots$

in $D_1(x)$ have $u_{01} \leftarrow u_{11} \leftarrow u_{21} \leftarrow \cdots$ ↯ ■

Measure & category

Defn: A (Borel) measure on X (st. Borel) is a function $\mu : \mathcal{B}(X) \to [0, \infty]$

s.t. ① $\mu(\phi) = 0$

② $A \subseteq B \Rightarrow \mu(A) \leq \mu(B)$

③ $(A_n)_{n \in \mathbb{N}}$ disjoint $\Rightarrow \mu(\bigsqcup A_n) = \sum \mu(A_n)$.

μ is probability if $\mu(X) = 1$.

$A \subseteq X$ (not necessarily Borel) is null if there is a Borel set $B \subseteq X$ s.t. $A \subseteq B$ and $\mu(B) = 0$.

A is conull if $X \setminus A$ is null.

$A \subseteq X$ is μ-measurable if there is a Borel set $B \subseteq X$ s.t. $A \triangle B$ is null.

Notes: ① Borel $\Rightarrow$ measurable

② measurable sets : σ-algebra generated by Borel sets + null sets.

③ For measurable A, can define $\mu(A) := \mu(B)$ for any (all) Borel B s.t. $A \triangle B = 0$

④ A is null $\iff$ A is measurable and $\mu(A)=0$.

A function $f: X \to Y$ is μ-measurable if f is equal to a Borel function on a conull set $\iff$ preimages of Borel sets are measurable.

←st. Borel (μ is on X)

Thm (regularity of measure): X Polish space, μ a finite measure on X. Then μ is regular, i.e., for any measurable A,

$\mu(X)<\infty$.

$$\mu(A) = \inf\{\mu(U) : U \supseteq A \text{ open}\}$$
$$= \sup\{\mu(F) : F \subseteq A \text{ closed}\}.$$

Corollary (99% lemma): μ finite measure on Polish X, $A \subseteq X$ measurable. If $\mu(A)>0$, then for any $\varepsilon>0$, can find open $U \supseteq A$ s.t. $\dfrac{\mu(A)}{\mu(U)} > 1-\varepsilon$.

Pf. Immediate.