Math 223D, Lecture 8

Thm (Regularity of measure): Let μ be a finite measure on a Polish X. ($\mu(X) < \infty$)
Then μ is regular, i.e., for all μ-measurable $A \subseteq X$,

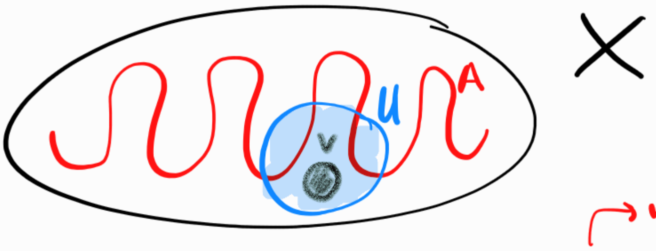
$$\mu(A) = \inf\{\mu(U) : U \supseteq A \text{ open}\}$$
$$= \sup\{\mu(F) : F \subseteq A \text{ closed}\}.$$

Corollary (99% lemma): μ a finite measure on Polish X, $A \subseteq X$ measurable. If A is not null, then for all $\varepsilon > 0$, there is open $U \supseteq A$ s.t. $\dfrac{\mu(A)}{\mu(U)} > 1 - \varepsilon$.

A is "most" of U

Baire category

X a top. space. A set $A \subseteq X$ is **nowhere dense** if $\overline{A}$ has $\varnothing$ interior; i.e., if for every open $\varnothing \neq U \subseteq X$, there is open $\varnothing \neq V \subseteq U$ s.t. $A \cap V = \varnothing$:



Nowhere dense sets form an ideal: (notion of smallness)

- $\varnothing$ is ND
- A is ND & $B \subseteq A \Rightarrow B$ is ND
- A, B ND $\Rightarrow A \cup B$ is ND.

But it is not a σ-ideal (not closed under ctbl unions;

e.g., $\mathbb{Q} = \bigcup \{q_n\} \subseteq \mathbb{R}$).

dense ND

Defn (Baire): A set is meager if it is a ctbl union of ND sets.

Meager sets form a σ-ideal ☺

But it could be useless: In $\mathbb{Q}$ with the subspace topology, every set is meager... ☹

not Polish!

Thm (Baire category): In a Polish space, non-$\varnothing$ open sets aren't meager.

comeager: complement of a meager set.

"large"

A property of pts in X is generic if it holds on a comeager set.

Comeager sets in a Polish space are dense (intersect every non-$\varnothing$ open set)

Intersection of ctbly many comeager sets is comeager.

Prop: A set $A \subseteq X$ in a Polish space is comeager
$\iff$ there are dense open sets $(U_n)_{n\in\mathbb{N}}$ s.t. $A \supseteq \bigcap_{n\in\mathbb{N}} U_n$
$\iff$ A has a dense G_δ subset.

Exercise: Prove this!

Example: μ = probability measure on $\mathcal{C}$, "fair coin-flips":

For $x \in \mathcal{C}$, $D \subset \mathbb{N}$ finite,

$$\mu(N_D(x)) = \frac{1}{2^{|D|}}.$$

probability of agreeing with x on D

$$H := \{x \in \mathcal{C} : \lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} x_i = \frac{1}{2}\}$$

Probability theory (law of large numbers): H is μ-conull.

But H is meager!

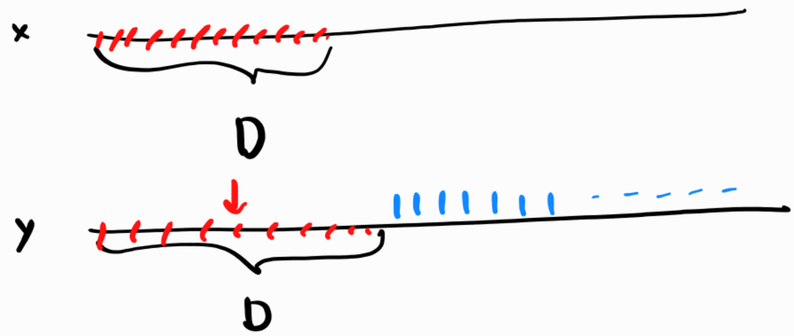
$$H^c \supset \{x \in \mathcal{C} : \underbrace{\forall N}_{\text{ctbl } \cap} \boxed{\underbrace{\exists n > N}_{\text{ctbl } \cup} \text{ s.t. } \underbrace{\frac{1}{n}\sum_{i=1}^{n-1} x_i > \frac{9}{10}}_{\substack{\text{finitely determined} \\ \Rightarrow \text{clopen}}}}\}.$$

open

$$U_N := \{x \in \mathcal{C} : \exists n > N \text{ s.t. } \frac{1}{n}\sum_{i<n} x_i > \frac{9}{10}\}$$

Density: Given $x \in \mathcal{C}$, $D \subset \mathbb{N}$ finite, want a pt $y \in U_N$ that agrees with x on D.

Set $y_i := \begin{cases} x_i & \text{if } i \in D, \\ 1 & \text{if } i \notin D. \end{cases}$



So H^c is comeager.

A set $A \subseteq X$ is Baire measurable (has the Baire property) if there is a Borel set B s.t. $A \triangle B$ is meager.

↳ topological version of measurability

Thm (100% lemma): X Polish, $A \subseteq X$ Baire measurable. Then there is an open set $U \subseteq X$ s.t. $A \triangle U$ is meager.

Cor (the Baire alternative): X Polish. Let $\mathcal{U}$ be a weak basis for X (i.e., non-ϕ open set has a non-ϕ subset from $\mathcal{U}$). If $A \subseteq X$ is Baire-measurable, then either

① A is meager

OR

② A is comeager in some non-$\emptyset$ $U \in \mathcal{U}$

(i.e., $U \setminus A$ is meager)

Example: Let $\mathbb{H}$ be the graph with vtx set $\mathcal{C}$ (unctbl!)

$\{x,y\} \in E(\mathbb{H}) \iff |x \triangle y| = 1$ (as subsets of $\mathbb{N}$)

∞-dimensional hypercube

Note: There is a path $x = x_0, x_1, x_2, \ldots, x_k = y$ from x to y in $\mathbb{H} \iff |x \triangle y|$ is finite.

$\Rightarrow$ $\mathbb{H}$ is disconnected (unctbly many components!)

Locally ctbl : every vtx has ctbly many nbrs

$\iff$ every component is ctbl

(exercise)

chromatic #

$\mathbb{H}$ is bipartite: No odd cycles! $\chi(\mathbb{H}) = 2$

$\exists\ \mathcal{C} = I_0 \cup I_1$ partition into $\mathbb{H}$-indep. sets.

Prop: There is no partition $\mathcal{C} = \bigcup_{n \in \mathbb{N}} I_n$ into ctbly many Borel $\mathbb{H}$-independent sets.

Borel chromatic # $\chi_B(\mathbb{H})$ is uncountable

TBC