Math 223D, Lecture 9

Example: Let $\mathbb{H}$ be the graph with vtx set $\mathcal{C}$ (unctbl!)

$\{x,y\} \in E(\mathbb{H}) \iff |x \triangle y| = 1$ (as subsets of $\mathbb{N}$)

∞-dimensional hypercube

Note: There is a path $x = x_0, x_1, x_2, \ldots, x_k = y$ from x to y in $\mathbb{H}$ $\iff$ $|x \triangle y|$ is finite.

$\Rightarrow$ $\mathbb{H}$ is disconnected (unctbly many components!)

Locally ctbl: every vtx has ctbly many nbrs

$\iff$ every component is ctbl

(exercise)

chromatic #

$\mathbb{H}$ is bipartite: No odd cycles! $\chi(\mathbb{H}) = 2$

$\exists\ \mathcal{C} = I_0 \cup I_1$ partition into $\mathbb{H}$-indep. sets.

> **Prop:** There is no partition $\mathcal{C} = \bigcup_{n \in \mathbb{N}} I_n$ into ctbly many Borel $\mathbb{H}$-independent sets.

$\hookrightarrow$ Borel chromatic # $\chi_B(\mathbb{H})$ is uncountable

fair coin-flip

In fact, μ-measurable chrom. # $\chi_\mu(\mathbb{H})$,

Baire-measurable chrom # $\chi_{BM}(\mathbb{H})$

are both unctbl as well

(and, of course, $\chi_B \geq \max\{\chi_\mu, \chi_{BM}\}$).

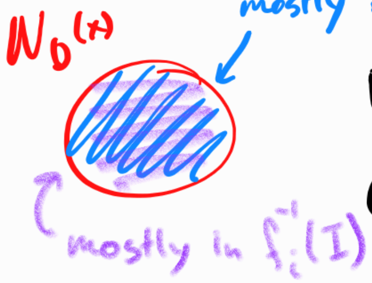
Pf: We claim that:

Every μ-meas. (Baire-meas.) $\mathbb{H}$-indep. set $I \subseteq \mathcal{C}$

is μ-null (meager)

Say $I \subseteq \mathbb{H}$ is BM, $\mathbb{H}$-indep., non-meager.

By the Baire alternative I is comeager in some non-ϕ basic open set, say $N_D(x)$ for some $x \in \mathcal{C}$, $D \subset_{\text{finite}} \mathbb{N}$



Pick any $i \in \mathbb{N} \setminus D$.

Consider $f_i : \mathcal{C} \to \mathcal{C}$

$f_i(y) :=$ "flip" the i^{th} coordinate

This is a homeomorphism, so it preserves topological notions

$\Rightarrow$ $f_i^{-1}(I)$ is comeager in $f_i^{-1}(N_D(x)) = N_D(x)$

$\Rightarrow$ $I \cap f_i^{-1}(I)$ is comeager in $N_D(x)$

$\Rightarrow$ $I \cap f_i^{-1}(I) \neq \phi$.

Take any $y \in I \cap f_i^{-1}(I)$. Then $y, f_i(y) \in I$.

But y and $f_i(y)$ are adjacent in $\mathbb{H}$! $\lightning$

μ-Measurable version: exercise (use the 99% lemma).

Graphs generated by functions

$T : X \to X$, no fixed pts (i.e., $Tx \neq x$ $\forall x \in X$).

Then define G_T : vtx set X, edge set $\{\{x, Tx\} : x \in X\}$

$\boxed{\text{Claim: } \chi(G_T) \leq 3}$ $\leftarrow$ optimal:

Proof: Use the following result:

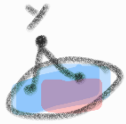
... Erdős): For finite $k \in \mathbb{N}$, a graph G

Thm (De Bruijn–Erdős): For [illegible]
is k-colorable $\iff$ every finite subgraph of G is k-colorable.

Proved using "compactness" (e.g., Tychonoff thm or compactness for first-order logic). (Uses AC)

Now, we need to check that every finite subgraph of G_T is 3-colorable. Suppose not & let $Y \subseteq X$ be a finite set of min size s.t. $G' := G_T[Y]$ is not 3-colorable.

$$\sum_{y \in Y} \deg_{G'}(y) = 2|E(G')| \leq 2|Y|$$ (since each $y \in Y$ is incident to ≤ 1 edge of the form $\{y, Ty\}$ in G')



$\Rightarrow$ G' has a vtx y of degree ≤ 2.

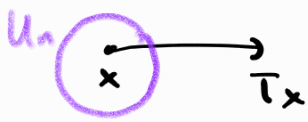
By the choice of Y, $G' - y$ is 3-colorable. Can extend the coloring to y $\lightning$

Now suppose X is standard Borel, $T: X \to X$ Borel.

What can we say about $\chi_B(G_T)$?

Prop (Kechris–Solecki–Todorcevic): $\chi_B(G_T)$ is <u>countable</u>.
I.e., there is a Borel proper coloring $c: X \to \mathbb{N}$.

Proof: Let $(U_n)_{n \in \mathbb{N}}$ be a ctbl basis for a compatible Polish topology on X. <u>Note</u>: For each $x \in X$, there is $n \in \mathbb{N}$ s.t.
$x \in U_n$, $Tx \notin U_n$



<u>Define</u>: $c(x) := \min\{n \in \mathbb{N} : x \in U_n,\ Tx \notin U_n\}$.

Clearly a proper coloring.

Borel: Need to show each $c^{-1}(n)$ is a Borel set.

Let $X_n := \{x \in X : x \in U_n,\ Tx \notin U_n\}$
$$= U_n \setminus T^{-1}(U_n) \quad \text{(Borel)}$$

Then $c^{-1}(n) = X_n \setminus (X_0 \cup \dots \cup X_{n-1})$ also Borel.

Example: The shift graph:

Let $X := \{x \in \mathcal{C} : x \text{ has } \infty\text{-many } 1\text{s}\}$

The shift operation: $S : X \to X$

$S(x) :=$ switch the first 1 to a 0.

$$S(\underbrace{\overset{0\,1\,2\,3\,4\,5\,6\,7\,8}{001011101}\dots}_{\Downarrow\ (2,4,5,6,8,\dots)}) = \underbrace{000011101\dots}_{\Downarrow\ (4,5,6,8,\dots)}$$

Note: G_S is locally finite (every vtx has $<\infty$ many nbrs),

G_S is a subgraph of $\mathbb{H} \Rightarrow \chi(G_S) = 2$,

G_S has no cycles (exercise), i.e., it is a forest.

Thm: $\chi_B(G_S) = \aleph_0$, i.e., no Borel coloring w/ finitely many colors.

↳ we'll prove this, but first...

Thm (KST): X st. Borel, $T : X \to X$ Borel, no fixed pts.

① If μ is a probability measure on X, then $\boxed{\chi_\mu(G_T) \le 3}$

② If τ is a compatible Polish top. on X, then $\boxed{\chi_{BM}(G_T) \le 3}$

So measure / Baire category can't* prove : the problems are always concentrated on a null / meager set!