Set-up: X a standard Borel space, $T : X \to X$ Borel function with no fixed pts.

G_T : vtx set X, edge set $\{\{x, Tx\} : x \in X\}$.

Know: $\chi(G_T) \leq 3$, $\chi_B(G_T) \leq \aleph_0$.

Thm: For any finite measure μ on X, $\boxed{\chi_\mu(G_T) \leq 3.}$
For any compatible Polish top. on X, $\boxed{\chi_{BM}(G) \leq 3.}$

A set $A \subseteq X$ is (forward-)recurrent (for T) if

$\forall x \in X, \exists k \in \mathbb{N}$ s.t. $T^k x \in A$

Let $R_A(x) := \min k \in \mathbb{N}$ s.t. $T^k x \in A$.

Note: $R_A(Tx) = R_A(x) - 1$ unless $x \in A$

Lemma: If there is a Borel recurrent G_T-independent set $I \subseteq X$, then $\chi_B(G_T) \leq 3$.

Pf: Define:

(Borel: exercise)

$$c(x) := \begin{cases} 0 & \text{if } x \notin I \text{ and } R_I(x) \text{ is even} \\ 1 & \text{if } x \notin I \text{ and } R_I(x) \text{ is odd} \\ 2 & \text{if } x \in I \end{cases}$$

Thm: TFAE:

① $\chi_B(G_T)$ is finite

② $\chi_B(G_T) \leq 3$

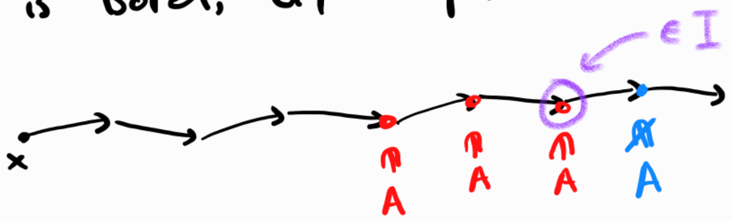
③ There is a Borel set $A \subseteq X$ s.t. both A and A^c are recurrent

recurrent.

Pf. ② ⇒ ① ✓

③ ⇒ ②: Set $I := \{x \in X : x \in A, Tx \notin A\} = A \setminus T^{-1}(A)$.

Then I is Borel, G_T-indep., and recurrent:



Hence, $\chi_B(G_T) \leq 3$ by the lemma.

① ⇒ ③: Let $f : X \to \{0, 1, \ldots, k-1\}$ be a finite Borel coloring.

Set $A := \{x \in X : f(x) > f(Tx)\}$, so

$$A^c := \{x \in X : f(x) < f(Tx)\}.$$

Then both A and A^c are recurrent.

Now let μ be a finite measure on X. We want to find a Borel set $A \subseteq X$ s.t. A, A^c are recurrent almost everywhere, i.e., the sets

$$\{x \in X : T^k x \in A \text{ for some } k \in \mathbb{N}\},$$
$$\{x \in X : T^k x \notin A \text{ for some } k \in \mathbb{N}\}$$

are μ-conull.

For $x \in X$ and $A \subseteq X$, let $N_A(x) := \{n \in \mathbb{N} : T^n x \in A\}$.

Lemma: There exists a Borel set $A \subseteq X$ s.t. for μ-almost all $x \in X$, the sets $N_A(x)$, $\underbrace{N_{A^c}(x)}_{= N_A(x)^c}$ are infinite.

Proof: Let $f : X \to \mathbb{N}$ be a Borel proper coloring of G_T.

What happens to our earlier construction?

$A_0 := \{x \in X : f(x) > f(Tx)\}$.

... A_0 itself may not be

Then A_0^c is still recurrent ... but A_0 itself may not be.

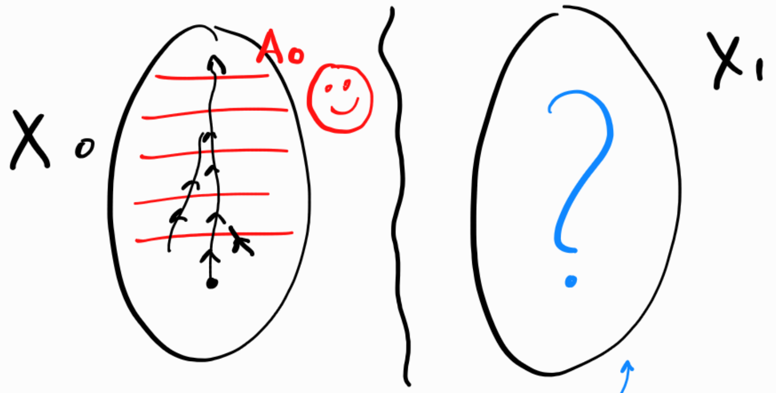
Partition X:

$$X_0 := \{x \in X : N_{A_0}(x) \text{ is infinite}\}$$

$$X_1 := \{x \in X : N_{A_0}(x) \text{ is finite}\} =$$

$$= \{x \in X : \text{the sequence } f(x), f(Tx), \dots \text{ is eventually increasing}\}$$

Note: Both X_0 and X_1 are T-invariant (closed under T).



Need to find a Borel set $A_1 \subseteq X_1$ s.t. for almost all $x \in X_1$, the sets $N_{A_1}(x)$, $N_{A_1^c}(x)$ are infinite (and then set $A = A_0 \cup A_1$).

From now on, work in X_1. I.e., assume $X_1 = X$.

$\rightarrow$ $\boxed{\text{For all } x \in X, \text{ the sequence } f(x), f(Tx), \dots \text{ is eventually increasing.}}$

Write $S(x) := \{f(T^n x) : x \in X\}$ $\leftarrow$ infinite subset of $\mathbb{N}$.

For $S \subseteq \mathbb{N}$, let $A_S := f^{-1}(S) = \{x \in X : f(x) \in S\}$.

Then $N_{A_S}(x), N_{A_S^c}(x)$ are infinite $\iff$ $\boxed{S(x) \cap S,\ S(x) \setminus S \text{ are infinite}}$

WANT to find $S \subseteq \mathbb{N}$ s.t. $S(x) \cap S$, $S(x) \setminus S$ are infinite for μ-almost all $x \in X$.

IDEA: Pick $S \subseteq \mathbb{N}$ randomly, by choosing each $n \in \mathbb{N}$ independently with prob. $\frac{1}{2}$.

Then, for each fixed $x \in X$,

$$\mathbb{P}[S(x) \cap S \text{ is } \infty] = \mathbb{P}[S(x) \setminus S \text{ is } \infty] = 1,$$

because $S(x)$ is infinite.

Thm (Fubini–Tonelli): $(X,\mu), (Y,\nu)$ finite measure spaces $f: X \times Y \to [0,\infty]$ $\mu \times \nu$-measurable. Then

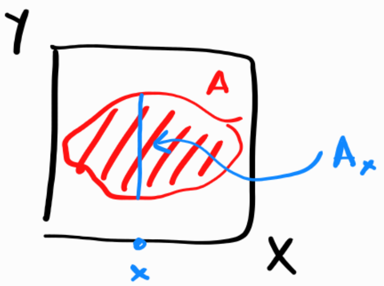
$$\int_{X \times Y} f(x,y)\, d(\mu\times\nu)(x,y) = \int_Y \left[\int_X f(x,y)\, d\mu(x)\right] d\nu(y) =$$

$$= \int_X \left[\int_Y f(x,y)\, d\nu(y)\right] d\mu(x).$$

Corollary: For $A \subseteq X \times Y$, let

$$A_x := \{y \in Y : (x,y) \in A\}$$

$$A^y := \{x \in X : (x,y) \in A\}$$



If A is $(\mu\times\nu)$-measurable, then

$$(\mu\times\nu)(A) = \int_X \nu(A_x)\, d\mu(x) = \int_Y \mu(A^y)\, d\nu(y).$$

In particular, the following are equivalent:

① A is $(\mu\times\nu)$-conull,

② for μ-almost all $x \in X$, A_x is ν-conull,

③ for ν-almost all $y \in Y$, A^y is μ-conull.

Now, let $(\mathcal{P}(\mathbb{N}), \nu)$ be the space of subsets of $\mathbb{N}$ $\cong \mathcal{C}$ the fair coin-flip measure. Define

$$A := \{(x,S) \in X \times \mathcal{P}(\mathbb{N}) : S(x) \cap S,\ S(x) \setminus S \text{ are infinite}\}$$

This is a Borel set (exercise) and we know that

for **all** $x \in X$, the set A_x is ν-conull.

By Fubini,

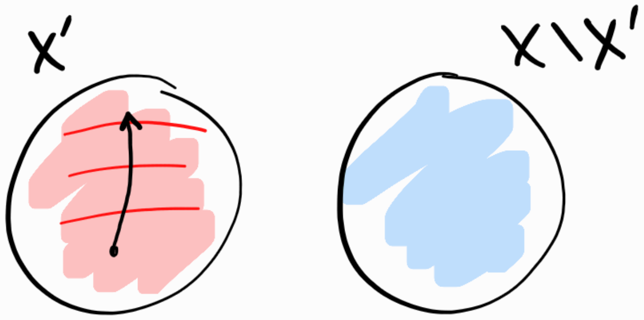
for **almost all** $S \subseteq \mathbb{N}$, the set A^S is μ-conull

Taking any $S \subseteq \mathbb{N}$ s.t. A^S is μ-conull finishes the proof. ■

Proof that $\chi_\mu(G_T) \le 3$: Take $A \subseteq X$ Borel s.t.

$X' := \{x \in X : N_A(x), N_{A^c}(x) \text{ are infinite}\}$ is μ-conull.

X' and $X \setminus X'$ are T-invariant. On X', A and A^c are recurrent $\Rightarrow \chi_B(G_T[X']) \le 3$.



Moreover, have a proper 3-coloring of $G_T[X \setminus X']$

Since $X \setminus X'$ is null, the union of these two colorings is measurable. ■

Proof that $\chi_{BM}(G_T) \le 3$: Same, except instead Fubini, use:

Thm (Kuratowski–Ulam): X, Y Polish spaces, $A \subseteq X \times Y$ Baire-measurable. The following are equivalent:

① A is comeager (in $X \times Y$)

① A is comeager (

② for generic $x \in X$, A_x is comeager (in Y)

③ for generic $y \in Y$, A^y is comeager (in X).

Fubini's thm: (X,μ), (Y,ν) finite measure spaces

$A \subseteq X \times Y$ measurable. Then

A is conull in $(X \times Y, \mu \times \nu)$ $\Leftrightarrow$

for μ-almost all $x \in X$, $\{y \in Y : (x,y) \in A\}$ is conull in (X,μ) $\Longleftrightarrow$

for ν-almost all $y \in Y$, $\{x \in X : (x,y) \in A\}$ is conull in (Y,ν).