Math 223D, Lecture 11

Last time: (X,μ), $T: X \to X$ Borel, no fixed pts

$\Rightarrow \chi_\mu(G_T) \leq 3$

Proof ideas: Start with a Borel coloring $c: X \to \mathbb{N}$

Hope: A Borel set $A \subseteq X$ s.t. both A and A^c are recurrent

$\boxed{\text{Case 1}}$: $A = \{x \in X : c(x) > c(Tx)\}$ works

$\boxed{\text{Case 2}}$: For all $x \in X$, the sequence $\{c(T^n x) : n \in \mathbb{N}\}$ is eventually increasing

Then take $A := \{x \in X : c(x) \in S\}$ for a random set $S \subseteq \mathbb{N}$

(works for almost all $x \in X$)

Tool: Fubini's thm: The statements

"for almost all x, for almost all y, $P(x,y)$" and

"for almost all y, for almost all x, $P(x,y)$"

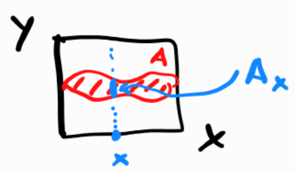
are equivalent (if the condition $P(x,y)$ is measurable).

To prove that when X is Polish, $\chi_{BM}(G_T) \leq 3$,

follow the same strategy, but replace Fubini with:

Thm (Kuratowski–Ulam): X, Y 2^{nd} ctbl top spaces,

$A \subseteq X \times Y$ Baire-measurable. Then TFAE:



① A is comeager in $X \times Y$

② The set $\{x \in X : A_x \text{ is comeager in } Y\}$ is comeager in X

(i.e., for generic $x \in X$, for generic $y \in Y$, $(x,y) \in A$)

③ For generic $y \in Y$, for generic $x \in X$, $(x,y) \in A$.

Exercise: Complete the proof of $\chi_{BM}(G_T) \leq 3$.

The Borel case:

For a set X, let $[X]^k := \{s \subseteq X : |s| = k\}$

$[X]^\infty := \{Y \subseteq X : Y \text{ is infinite}\}$

$[\mathbb{N}]^\infty \cong \{x \in \mathcal{C} : x \text{ has } \infty \text{ many } 1\text{s}\}$ (Polish)

$$S : [\mathbb{N}]^\infty \to [\mathbb{N}]^\infty : X \mapsto X \setminus \{\min X\}.$$

$$\boxed{\textbf{Thm: } \chi_B(G_S) = \aleph_0}$$

Proof uses (infinite-dimensional) Ramsey theory

Pigeonhole Principle: Let $q \in \mathbb{N}$. If $f : \mathbb{N} \to q$, then there is $X \in [\mathbb{N}]^\infty$ s.t. f is constant on X.

Ramsey's thm: Let $q, d \in \mathbb{N}$. If $f : [\mathbb{N}]^d \to q$, then there is $X \in [\mathbb{N}]^\infty$ s.t. f is constant on $[X]^d$. "d-dimensional" version of Pigeonhole

Proof idea: Induction on d.

What if $f : [\mathbb{N}]^\infty \to q$? There may not be $X \in [\mathbb{N}]^\infty$ s.t. f is constant on $[X]^\infty$ ☹ E.g., let $f : [\mathbb{N}]^\infty \to \{0,1\}$ be a proper 2-coloring of G_S. Then for all $X \in [\mathbb{N}]^\infty$,

$$f(X) \neq f(S(X)) = f(X \setminus \{\min X\}).$$

Thm (Galvin-Prikry): Let $q \in \mathbb{N}$. If $f : [\mathbb{N}]^\infty \to q$ is Borel, then there is $X \in [\mathbb{N}]^\infty$ s.t. f is constant on $[X]^\infty$.

$\Rightarrow$ G_S has no proper Borel q-coloring for any $q \in \mathbb{N}$, hence $\chi_B(G_S) = \aleph_0$, as claimed.

Since for $X \in [\mathbb{N}]^\infty$, $[X]^\infty \cong [\mathbb{N}]^\infty$, it is enough to prove the thm for $q=2$ (and then apply induction on q)

A set $\mathcal{U} \subseteq [\mathbb{N}]^\infty$ is **Ramsey** if for all $X \in [\mathbb{N}]^\infty$, there is $X' \in [X]^\infty$ s.t. either $[X']^\infty \subseteq \mathcal{U}$ or $[X']^\infty \cap \mathcal{U} = \emptyset$.

So the Galvin-Prikry thm is equivalent to:

Thm: Every Borel set $\mathcal{U} \subseteq [\mathbb{N}]^\infty$ is Ramsey.

Proof uses complicated induction $\rightsquigarrow$ need a stronger hypothesis

Defn: Use lowercase letters $a, b, \ldots, x, y, \ldots$ for finite subsets of $\mathbb{N}$, uppercase $A, B, \ldots, X, Y, \ldots$ for infinite ones.

Given a, A, write

$$\underbrace{[a}_{\text{must take}}, \underbrace{A]}_{\text{may take}} := \{ X \in [\mathbb{N}]^\infty : a \subseteq X \subseteq A \cup a \}.$$

E.g., $[\emptyset, A] = [A]^\infty$.

A set $\mathcal{U} \subseteq [\mathbb{N}]^\infty$ is **completely Ramsey** if for all a, A, there is $B \in [A]^\infty$ s.t. either $[a, B] \subseteq \mathcal{U}$ or $[a, B] \cap \mathcal{U} = \emptyset$.

completely Ramsey $\Rightarrow$ Ramsey

We'll prove:

Thm: Every Borel set $\mathcal{U} \subseteq [\mathbb{N}]^\infty$ is completely Ramsey.

And more!

Note: $[a,A] \cap [b,B] =$ either $\emptyset$ or $[a \cup b, A \cap B]$

Therefore, $\{ [a,A] : a \in [\mathbb{N}]^{<\infty}, A \in [\mathbb{N}]^{\infty} \}$ is a basis of a topology on $[\mathbb{N}]^{\infty}$, called the Ellentuck topology.

Note: ① Not Polish!

② Finer than the standard topology: every fin. det. set $\mathcal{U} \subseteq [\mathbb{N}]^{\infty}$ is Ellentuck-open (exercise). Hence, Borel sets in the standard topology are Ellentuck-Borel.

Thm (Ellentuck): A set $\mathcal{U} \subseteq [\mathbb{N}]^{\infty}$ is completely Ramsey $\iff$ $\mathcal{U}$ is Baire-measurable in the Ellentuck topology.