Math 223D, Lecture 12

$[\mathbb{N}]^{<\infty}$ ← finite subsets, $a, b, \dots, x, y, \dots$

$[\mathbb{N}]^{\infty}$ ← infinite subsets, $A, B, \dots, X, Y, \dots$

$[a, A] := \{ X \in [\mathbb{N}]^{\infty} : a \subseteq X \subseteq a \cup A \} = [a, a \cup A]$

A set $\mathcal{U} \subseteq [\mathbb{N}]^{\infty}$ is completely Ramsey if for all a, A, there is $X \subseteq A$ s.t. either $[a, X] \subseteq \mathcal{U}$ or $[a, X] \cap \mathcal{U} = \emptyset$.

Ellentuck topology on $[\mathbb{N}]^{\infty}$: generated by the sets $[a, A]$.

Thm (Ellentuck ⇒ Galvin–Prikry): A set $\mathcal{U} \subseteq [\mathbb{N}]^{\infty}$ is completely Ramsey $\iff$ $\mathcal{U}$ is Baire-measurable in the Ellentuck topology.

easy direction

Proof of $\Rightarrow$:

Suppose $\mathcal{U}$ is completely Ramsey. Define

$$\mathcal{V} := \bigcup \{ [a, A] : [a, A] \subseteq \mathcal{U} \} \subseteq \mathcal{U}$$

$\mathcal{V}$ is Ellentuck-open. We claim $\mathcal{U} \setminus \mathcal{V}$ is Ellentuck-nowhere-dense (hence meager), so $\mathcal{U}$ is Baire-measurable, as desired.

Indeed, take any basic open $[a, A]$.
$\mathcal{U}$ is completely Ramsey, so we have 2 cases:

Case 1: There is $X \subseteq A$ s.t. $[a, X] \subseteq \mathcal{U}$.
Then $[a, X] \subseteq \mathcal{V}$, so $[a, X]$ is an open subset of $[a, A]$ disjoint from $\mathcal{U} \setminus \mathcal{V}$. ✓

Case 2: There is $X \subseteq A$ s.t. $[a, X] \cap \mathcal{U} = \emptyset$. Then, again, $[a, X]$ is an open subset of $[a, A]$ disjoint from $\mathcal{U} \setminus \mathcal{V}$. ∎

Now for the hard direction: Ellentuck-BM ⇒ completely Ramsey.

Notation: For $A, B \in [\mathbb{N}]^{\infty}$, write

$A =^* B$ if $A \triangle B$ is finite (almost equality)

$A \subseteq^* B$ if $A \setminus B$ is finite (almost inclusion)

$A \subseteq^* B$ if $A \setminus B$ is finite

Lemma (diagonal intersection): If $A_0 \supseteq^* A_1 \supseteq^* A_2 \supseteq \cdots$, then there is $A \in [\mathbb{N}]^\infty$ s.t. $A \subseteq^* A_i$ for all $i \in \mathbb{N}$. (May arrange $A \subseteq A_0$)

Pf: Recursively pick a_i from $\underbrace{A_0 \cap \cdots \cap A_i}_{\text{infinite}}$ distinct from $a_0, \ldots, a_{i-1}$.

Let $A := \{a_0, a_1, \ldots\}$

Now, fix $\mathcal{U} \subseteq [\mathbb{N}]^\infty$.

Given $a \in [\mathbb{N}]^{<\infty}$, $A \in [\mathbb{N}]^\infty$, say:

- A accepts a if $\exists A' =^* A$ s.t. $[a, A'] \subseteq \mathcal{U}$
  ↳ may assume $A' \subseteq A$
- A rejects a if $\forall B \subseteq^* A$, $[a, B] \not\subseteq \mathcal{U}$.
  enough to consider $B \subseteq A$

$\Updownarrow$

$\forall B \subseteq^* A$, B does not accept a.

Observations:

① A accepts/rejects a, $B \subseteq^* A \Rightarrow B$ also accepts/rejects a.

② $\forall a, A$, there is $B \subseteq A$ s.t. B either accepts or rejects a.

③ $\forall A$, there is $D \subseteq A$ s.t. D either accepts or rejects every $a \in [\mathbb{N}]^{<\infty}$.
↳ call such set D decisive.

Proof. Enumerate all finite subsets of $\mathbb{N}$: $a_0, a_1, a_2, \ldots$

Iteratively apply ② to produce $A \supseteq A_0 \supseteq A_1 \supseteq \cdots$

s.t. A_i either accepts or rejects a_i.

Take $D :=$ diagonal intersection of $(A_i)_{i \in \mathbb{N}}$.

④ If A rejects a, then the set

$\{n : A \ldots A$ accepts $a \cup \{n\}\}$ is finite.

$N := \{ n \in A :$ A accep…

Proof: Suppose N is infinite. Recursively define elements $n_i \in N$ and finite sets $f_i \subseteq A$ as follows: Once $n_0, \dots, n_{i-1}, f_0, \dots, f_{i-1}$ are defined, pick $n_i \in N \setminus (f_0 \cup \dots \cup f_{i-1})$ distinct from $n_0, \dots, n_{i-1}$. Since A accepts $a \cup \{n_i\}$, there is a finite set f_i s.t.

$$[a \cup \{n_i\}, A \setminus f_i] \subseteq \mathcal{U}.$$

Now, consider $B := \{n_0, n_1, \dots\} \in [N]^\infty$.

We claim that $[a, B] \subseteq \mathcal{U}$, contradicting the assumption that A rejects a. Indeed, take any $a \subseteq C \subseteq a \cup B$. Say i is minimum s.t. $n_i \in C$. Then

$$a \cup \{n_i\} \subseteq C \subseteq a \cup \{n_i\} \cup (A \setminus f_i),$$

hence $C \in [a \cup \{n_i\}, A \setminus f_i] \subseteq \mathcal{U}$, as claimed. $\blacksquare$

⑤ If D is decisive, then

- **either** D accepts $\emptyset$
- **or** there is $E \in [D]^\infty$ s.t. $\boxed{D \text{ rejects}}$ all $a \subset E$.

hence E also rejects all $a \subset E$.

Proof: Suppose D rejects $\emptyset$. Recursively construct finite sets $\emptyset = e_0 \subset e_1 \subset \cdots$, subsets of D, s.t. D rejects every subset of each e_i.

$e_0 = \emptyset$ has this property by assumption.

Given e_{i-1}, there are only finitely many $n \in \mathbb{N}$ s.t. D accepts $a \cup \{n\}$ for some $a \subseteq e_{i-1}$ (by ④), so we can pick $n_i \in D \setminus e_{i-1}$ s.t. D rejects all subsets of

$e_i := e_{i-1} \cup \{n_i\}$.

Now we set $E := \bigcup_{i \in \mathbb{N}} e_i$.