Math 223D, Lecture 13

Recall: $A =^* B$ if $A \triangle B$ is finite; $A \subseteq^* B$ if $A \setminus B$ is finite

Fix $\mathcal{U} \subseteq [\mathbb{N}]^\infty$. Given $a \in [\mathbb{N}]^{<\infty}$, $A \in [\mathbb{N}]^\infty$:

- A **accepts** a if $\exists A' =^* A$ s.t. $[a, A'] \subseteq \mathcal{U}$
- A **rejects** a if $\forall B \subseteq^* A$, $[a, B] \not\subseteq \mathcal{U}$

  $\iff \forall B \subseteq A$, B does not accept a.

Last time:

① A accepts/rejects a, $B \subseteq^* A \Rightarrow B$ also accepts/rejects a.

② $\forall A$, there is $D \subseteq A$ s.t. $\underbrace{D \text{ either accepts or rejects } \textbf{every } a \in [\mathbb{N}]^{<\infty}}$.

call such set D **decisive**.

③ If A rejects a, then the set

$N := \{ n \in A : A \text{ accepts } a \cup \{n\} \}$ is finite.

Final observation:

④ If D is decisive, then

either D accepts $\emptyset$

or there is $X \in [D]^\infty$ s.t. $\boxed{D \text{ rejects}}$ all $x \subset X$. hence X also rejects all $a \subset E$.

Proof: Suppose D rejects $\emptyset$. Recursively construct finite sets $\emptyset = x_0 \subset x_1 \subset \cdots$, subsets of D, s.t.

D rejects every subset of each x_i.

$x_0 = \emptyset$ has this property by assumption.

Given x_{i-1}, there are only finitely many $n \in \mathbb{N}$ s.t.

D accepts $a \cup \{n\}$ for some $a \subseteq x_{i-1}$ (by ③), so we can pick $n_i \in D \setminus X_{i-1}$ s.t. D rejects all subsets of $x_i := x_{i-1} \cup \{n_i\}$.

Now we set $X := \bigcup_{i \in \mathbb{N}} x_i$.

We are ready to prove Galvin-Prikry for open sets:

Proposition: Every Ellentuck-open set $\mathcal{U} \subseteq [\mathbb{N}]^\omega$ is completely Ramsey.

Proof: We'll prove $\mathcal{U}$ is Ramsey, leaving "completely" as an exercise.

Take $A \in [\mathbb{N}]^\omega$. **Want:** $X \in [A]^\omega$ s.t. either $[X]^\omega \subseteq \mathcal{U}$ or $[X]^\omega \cap \mathcal{U} = \phi$.

By ②, have a decisive $D \subseteq A$.

Case 1: D accepts $\phi \Rightarrow \exists X \subseteq D$ co-finite s.t. $[\phi, X] = [X]^\omega \subseteq \mathcal{U}$ ✓

Case 2: D rejects ϕ

By ④, have $X \subseteq D$ s.t. X rejects all $x \subset X$.

We claim that $[X]^\omega \cap \mathcal{U} = \phi$, as desired.

Otherwise, there is $Y \subseteq X$ s.t. $Y \in \mathcal{U}$.

$\mathcal{U}$ is Ellentuck-open, so there is $x \subset Y$ s.t. $\overbrace{[x, Y]}^{\text{basic open nbhd of } Y.} \subseteq \mathcal{U}$

But then X accepts x! ↯

Next we consider nowhere dense sets

Proposition: If $\mathcal{F} \subseteq [\mathbb{N}]^\omega$ is nowhere-dense (in the Ellentuck top) then for all a, A, there is $X \subseteq A$ s.t. $[a, X] \cap \mathcal{F} = \phi$.

Moreover, given $b \subset A$, can find $X \subseteq A$ s.t. $b \subset X$ and $[a, X] \cap \mathcal{F} = \phi$.

Proof: May assume $\mathcal{F}$ is closed with empty interior (replace $\mathcal{F}$ with $\bar{\mathcal{F}}$)

Then $\mathcal{F}^c$ is open $\Rightarrow \mathcal{F}^c$ is completely Ramsey $\Rightarrow \mathcal{F}$ is comp. Ramsey.

Hence, $\exists X \subseteq A$ s.t. either $[a, X] \cap \mathcal{F} = \phi$ (and we are done) or $[a, X] \subseteq \mathcal{F}$ $\rightsquigarrow$ so $\mathcal{F}$ has non-$\emptyset$ interior ↯

For the "moreover" part: Apply the previous argument iteratively to find $Y \subseteq A$ s.t. $[a \cup c, Y] \cap \mathcal{F} = \emptyset$ for all $c \subseteq b$.

Then let $X := b \cup Y$.

Proposition: Meager sets in the Ellentuck topology are nowhere-dense.
(hence completely Ramsey!)

Pf: Suppose $(\mathcal{F}_i)_{i \in \mathbb{N}}$ are nowhere-dense and consider

$\mathcal{F} := \bigcup_{i \in \mathbb{N}} \mathcal{F}_i$. Take any basic open, $[a, A]$.

Want: A non-$\emptyset$ open subset of $[a, A]$ disjoint from $\mathcal{F}$.

Recursively pick elts $n_0, n_1, \dots \in A$ and subsets $A \supseteq A_0 \supseteq A_1 \supseteq \cdots$ so that $n_0, n_1, \dots, n_i \in A_i$ and $[a, A_i] \cap \mathcal{F}_i = \emptyset$.

Then let $X := \{n_0, n_1, \dots\} \subseteq \bigcap_i A_i$ and note that

$[a, X] \cap \mathcal{F}_i = \emptyset$ for all i, as desired.

Since completely Ramsey sets form a Boolean algebra, this shows that all Baire-measurable sets in the Ellentuck topology are completely Ramsey