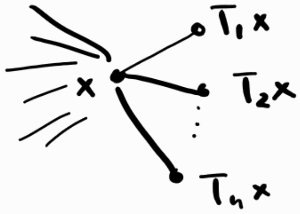
Math 223D, Lecture 14

$\boxed{n \in \mathbb{N}}$

Graphs generated by multiple functions:

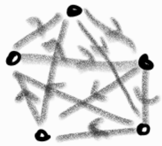
Let $T_1, \dots, T_n : X \to X$ be fixed-pt-free. Define

$$G_{T_1,\dots,T_n} := G_{T_1} \cup \dots \cup G_{T_n}.$$



Note: $\chi(G_{T_1,\dots,T_n}) \leq 2n+1$

sharp: K_{2n+1}



Assume X a st. Borel space, $T_1, \dots, T_n$ Borel maps.

Lemma: $\chi_B(G_{T_1,\dots,T_n}) \leq \aleph_0$.

Pf. Let $c_i : X \to \mathbb{N}$ be a Borel proper coloring of G_{T_i}.

Then $c : X \to \mathbb{N}^n : x \mapsto (c_1(x), \dots, c_n(x))$ is a Borel proper coloring of $G_{T_1,\dots,T_n}$. ctbl ∎

Lemma: If $\chi_B(G_{T_1,\dots,T_n}) < \aleph_0$, then $\chi_B(G_{T_1,\dots,T_n}) \leq 3^n$.

Pf: $\chi_B(G_{T_1,\dots,T_n}) < \aleph_0 \Rightarrow \chi_B(G_{T_i}) < \aleph_0 \Rightarrow \chi_B(G_{T_i}) \leq 3$

$c_i : X \to \{0,1,2\}$ Borel proper coloring of G_{T_i}. Then

$c : X \to \{0,1,2\}^n : x \mapsto (c_1(x), \dots, c_n(x))$ is a Borel 3^n-coloring of $G_{T_1,\dots,T_n}$. ∎

Can do better!

Say $f : X \to q \in \mathbb{N}$ is a Borel q-coloring of $\chi_B(G_{T_1,\dots,T_n})$.

Recursively define $c : X \to \{0,1,\dots,n\}$:

$$c(x) := \min\{k : c(T_i x) \neq k \text{ for all } i \text{ s.t. } f(T_i x) > f(x)\}$$

Look at the values of f

well-defined and Borel

well-defined
(determined by the values of f on pts of the form $T_{i_1} T_{i_2} \cdots T_{i_r} x$, $r \le q$)

By definition, if $f(x) < f(T_i x)$, then $c(x) \neq c(T_i x)$.
But if $f(x) > f(T_i x)$, may have $c(x) = c(T_i x)$ ....
Define (also recursively) $c' : X \to \{0, 1, \dots, n\}$ by:

$$c'(x) := \min \{ k : c'(T_i x) \neq k \text{ for all } i \text{ s.t. } c(x) = c(T_i x) \}.$$

implies $f(x) > f(T_i x)$

Again, well-defined & Borel.

Now, $x \mapsto (c(x), c'(x)) \in \{0, 1, \dots, n\}^2$ is a Borel proper coloring, so $\chi_B(G_{T_1, \dots, T_n}) \leqslant (n+1)^2$.

Can do even better!

Notice that if $c(x) = k$, then there are $\geqslant k$ indices i s.t. $\underline{f(T_i x) > f(x)}$.

On the other hand, if $c'(x) = k$, there are $\geqslant k$ indices i s.t. $c(x) = c(T_i x) \Rightarrow \underline{f(T_i x) < f(x)}$.

$$\Longrightarrow \boxed{c(x) + c'(x) \leq n}$$

So the pair $(c(x), c'(x))$ can take at most $\binom{n+2}{2}$ values.

Thus:

$$\boxed{\text{If } \chi_B(G_{T_1, \dots, T_n}) < \aleph_0, \text{ then } \chi_B(G_{T_1, \dots, T_n}) \leq \binom{n+2}{2}.}$$

For $n = 2$: lower bound: $2 \cdot 2 + 1 = 5$
upper bound: $(2+2$ ... 6

upper bound: $\binom{\ }{2} = 6$

Thm (Palamourdas (UCLA :))): If $\chi_B(G_{T_1,T_2}) < \aleph_0$, then $\chi_B(G_{T_1,T_2}) \leq 5$ $\longrightarrow$ HW! :)

7 or 8

OPEN: What is the right bound for $n = 3, 4, 5, \ldots$? If it linear in n? Quadratic? ¯_(ツ)_/¯

General theory of Borel graphs

A **graph** G: vtx set $V(G)$, edge set $E(G) \subseteq [V(G)]^2$

↳ undirected

Defn: A Borel (analytic, etc.) graph G is a graph s.t. $V(G)$ is a standard Borel space,

$$\{(x,y) \in V(G)^2 : \{x,y\} \in E(G)\}$$

is a Borel (analytic, etc.) subset of $V(G)^2$.

Spaces of finite sets

Let X be a standard Borel space, $k \in \mathbb{N}$. Want to make $[X]^k$ a standard Borel space as well.

(I) Let $(X)^k := \{(x_1, \ldots, x_k) \in X^k : x_1, \ldots, x_k \text{ are distinct}\}$

This is a Borel subset of X^k (so a standard Borel space)

Consider the map $\pi : (X)^k \longrightarrow [X]^k : (x_1, \ldots, x_k) \mapsto \{x_1, \ldots, x_k\}$.

Define a set $A \subseteq [X]^k$ to be **Borel** if $\pi^{-1}(A) \subseteq (X)^k$ is Borel.

(quotient structure)

This makes $[X]^k$ a st. Borel space.

(II) Fix a compatible complete metric d on X.

For $\{x_1, \ldots, x_k\}, \{y_1, \ldots, y_k\} \in [X]^k$, let

$$d(\{x_1,\dots,x_k\},\{y_1,\dots,y_k\}) := \min_{\sigma} \max_{i} d(x_i, y_{\sigma i})$$

↳ permutation of $\{1,\dots,k\}$

<u>Note</u>: For $k \geq 2$, this metric is not complete, but it extends to a complete metric on $[X]^{\leq k}$, where $[X]^k$ is an open subset

This makes $[X]^k$ a Polish space and yields the same st. Borel structure as in I.

III Fix a Borel linear order $<$ on X (exists by the Borel iso. theorem: $X \cong \mathbb{R}$ if it is unctbl)

↳ as a subset of X^2

Then identify $[X]^k$ with the Borel set

$$\{(x_1,\dots,x_k) \in X^k : x_1 < x_2 < \dots < x_k\}.$$

This again gives $[X]^k$ the same Borel σ-algebra as in I, II.

Now we can say:

A graph G is Borel (analytic, etc.) if $V(G)$ is a st. Borel space, $E(G)$ is a Borel (analytic, etc.) subset of $[V(G)]^2$.