Math 223D, Lecture 15

2-coloring dichotomy: For any graph G,

either $\chi(G) \le 2$

or G contains an odd cycle

↳ concrete obstruction to 2-coloring

There is a version of this for ctbl Borel chromatic #:

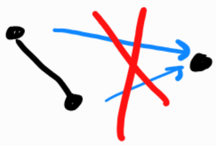
Thm (Kechris-Solecki-Todorcevic, G_0-dichotomy):
There exists a Borel graph G_0 s.t. for every analytic graph G on a Polish space, exactly one of the following holds:

① $\chi_B(G) \le \aleph_0$ $\quad (G \to_B K_{\aleph_0})$

② there is a continuous homomorphism from G_0 to G. $\quad G_0 \to_c G$

A homomorphism from H to G: a map $f: V(H) \to V(G)$ that sends edges of H to edges of G, i.e., if $\{x,y\} \in E(H)$, then $\{f(x), f(y)\} \in E(G)$.

Note: A proper coloring $\Leftrightarrow$ a homomorphism to a complete graph



Write: $H \to_B G$ if there is a Borel homomorphism from H to G

$H \to_c G$ if — " — " continuous — " — " — " —

Then $H \to_c G \Rightarrow H \to_B G \Rightarrow \chi_B(H) \le \chi_B(G)$

Construction of G_0

Study certain subgraphs of $\mathbb{H}$.

Let $2^* = \{0,1\}^* := \{$ all finite strings of 0s and 1s $\}$.

For $S \subseteq 2^*$

For $S \subseteq 2^*$, let $\mathbb{G}_S$ be the spanning subgraph of $\mathbb{H}$ with

$V = \mathcal{C}$

edges of the form:

$$\{ s^\frown 0^\frown t,\ s^\frown 1^\frown t \} \quad \text{for } s \in S,\ t \in \mathcal{C}.$$

(here $^\frown$ denotes concatenation)

Examples:

- $\mathbb{H} = \mathbb{G}_{2^*}$
- $\mathbb{G}_\varnothing$ has no edges
- $\mathbb{G}_{\{\varnothing, 0, 00, 000, 0000, \ldots\}}$ = the shift graph + the pts of $\mathcal{C}$ with finitely many 1s.

Properties of $\mathbb{G}_S$:

Fact 1: If $S \subseteq 2^*$ contains at most one string of each finite length, then $\mathbb{G}_S$ is a forest (no cycles).

Fact 2: If $S \subseteq 2^*$ contains at least one string of each finite length, then for any $x, y \in \mathcal{C}$,

there is a path from x to y in $\mathbb{G}_S$

$\iff$ there is a path from x to y in $\mathbb{H}$

$\iff$ $|x \triangle y|$ is finite.

Proofs: HW5 ☺

So, if S contains exactly one string of each finite length, then $\mathbb{G}_S$ is a spanning forest of $\mathbb{H}$.

Defn: A set $S \subseteq 2^*$ is dense if for $t \in 2^*$, there is some $s \in S$ s.t. $t \sqsubseteq s$ (t is a prefix of s).

Note: There exists a dense set $S \subseteq 2^*$ that includes exactly one string of each finite length:

$\varnothing$

0

1 0

00 0

01 00

10 000

11 0000

000 0000

001 00000

010 000000

011 0000000

... .

Proposition: If $S \subseteq 2^*$ is dense, then every G_S-independent Baire-measurable set $I \subseteq \mathcal{C}$ is meager; hence $\chi_B(G_S) = \chi_{BM}(G_S) = 2^{\aleph_0}$.

↳ If S is dense and includes one string of each length, the graph G_S has no cycles and unctbl Borel χ.

↳ this is G_0

Proof: Suppose $I \subseteq \mathcal{C}$ is Baire-measurable and not meager. For each $t \in 2^*$, let $U_t := \{x \in \mathcal{C} : t \sqsubset x\}$ clopen

These sets form a basis for the topology on $\mathcal{C}$ (exercise).

Hence, by the Baire alternative, there is $t \in 2^*$ s.t. I is comeager in U_t. Because S is dense, we can find

$s \in S$ s.t. $t \sqsubseteq s \Rightarrow U_s \subseteq U_t$

$\Rightarrow I$ is comeager in U_s.

Now let $\ell := \text{length}(s)$ and define

$f : \mathcal{C} \to \mathcal{C}$ by $f(x) :=$ "flip" the ℓ^{th} bit.

Then f is a homeomorphism of $\mathcal{C}$, so $f^{-1}(I)$ is comeager in $f^{-1}(U_s) = U_s \Rightarrow I \cap f^{-1}(I)$ is comeager in U_s

$\Rightarrow$ there is a pt $x \in I \cap f^{-1}(I) \cap U_s$.

This means that $f(x) \in I$. But x and $f(x)$ are adjacent

... $f(x) \in I$. But x and $f(x)$ are adjacent in G_S, hence I is not G_S-independent. ■

Thm: If $S \subseteq 2^*$ contains exactly one string of each finite length, then, for every analytic graph G on a Polish space,

$$\chi_B(G) = 2^{\aleph_0} \implies G_S \to_c G.$$

This implies the G_0-dichotomy by taking $G_0 = G_S$ for S dense and as above.

Proof idea: $S = \{s_0, s_1, s_2, \ldots\}$ where $\text{length}(s_i) = i$ $\quad s_0 = \varnothing$

"Approximate" G_S by finite graphs $G_0, G_1, G_2, \ldots$

G_n: vertex set is $\{0,1\}^n$

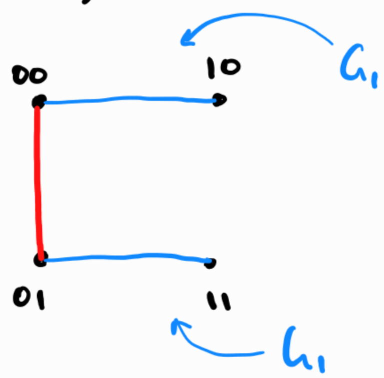
edges of the form $\{s_i{}^\frown 0^\frown t,\ s_i{}^\frown 1^\frown t\}$ where $i < n$ and $t \in 2^*$, $\text{length}(t) = n-i-1$.

Example: $S' = \{\varnothing, 0, 01, 101, \ldots\}$

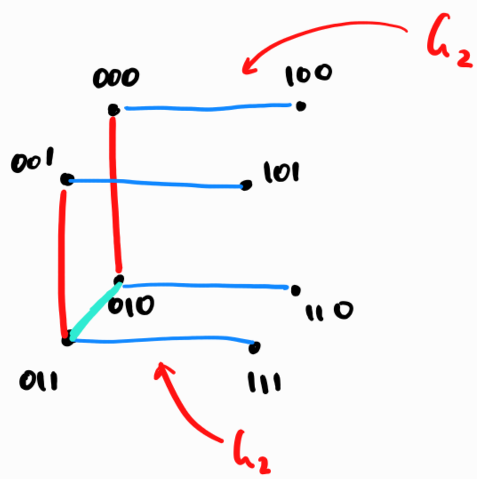
G_0:

G_1:

G_2:



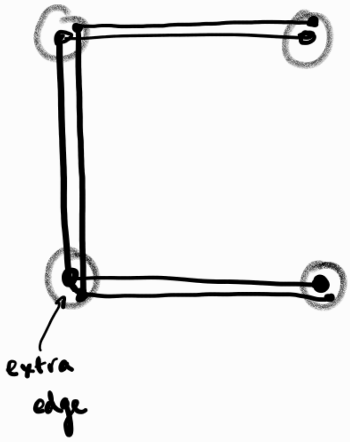
G_3:



And so on....

G_{n+1} is obtained from two copies of G_n by adding single edge between them joining $s_n{}^\frown 0$ and $s_n{}^\frown 1$.

We will build a continuous homomorphism $G_S \longrightarrow G$ as a "limit" of homomorphisms $G_n \longrightarrow G$ as $n \longrightarrow \infty$.



TBC!