Math 223D, Lecture 16

Recall: For $S \subseteq 2^* = \{\text{finite bit strings}\}$, let $\mathbb{G}_S$ be the graph on $\mathcal{C}$ with edges of the form

$$\{ s^\frown 0^\frown t,\ s^\frown 1^\frown t \},\quad s \in S,\ t \in \mathcal{C}.$$

$\rightsquigarrow$ a spanning subgraph of $\mathbb{H}$.

- If S contains exactly one string of each finite length, $\mathbb{G}_S$ is a spanning subforest (maximal acyclic subgraph) of $\mathbb{H}$.

S is **dense** if for all $t \in 2^*$, there is $s \in S$ s.t. $t \sqsubseteq s$.

- If S is dense, then every $\mathbb{G}_S$-indep. Baire-meas. set is meager, hence $\chi_B(\mathbb{G}_S) = \chi_{BM}(\mathbb{G}_S) = 2^{\aleph_0}$.

Thm ($\mathbb{G}_0$-dichotomy): If $S \subseteq 2^*$ contains exactly one string of each length and G is an analytic graph on a Polish space with $\chi_B(G) = 2^{\aleph_0}$, then $\mathbb{G}_S \to_c G$.

continuous homomorphism

$\rightsquigarrow$ For dense S, such $\mathbb{G}_S$ (often denoted $\mathbb{G}_0$) is a "concrete reason" for the lack of a countable Borel coloring of G.

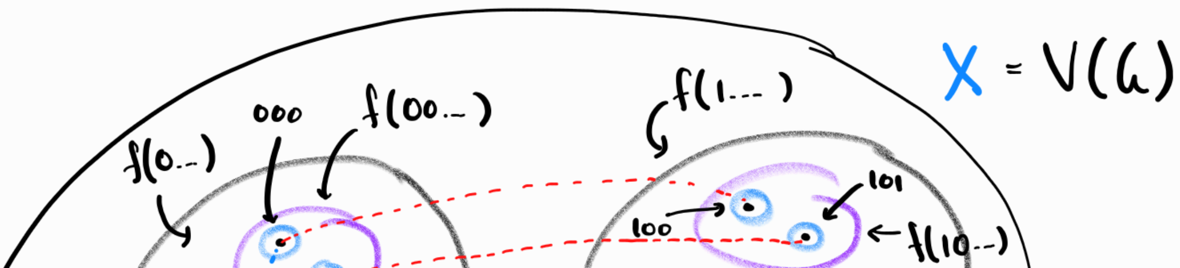
Proof cartoon:

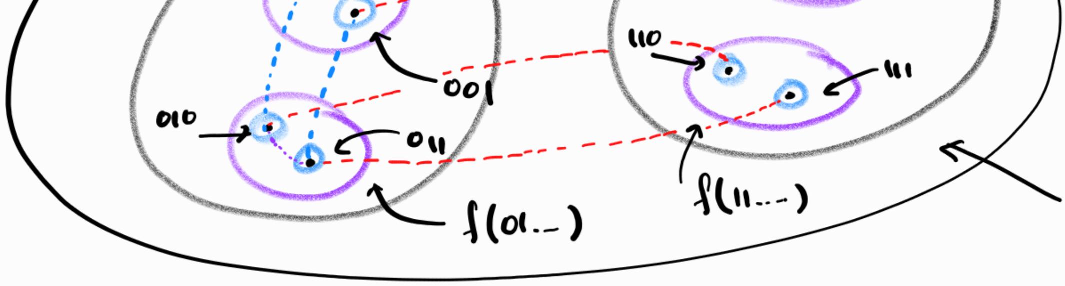
Fix $S \subseteq 2^*$ with exactly one string of each length.

Let H_n be the finite graph with vtx set $\{0,1\}^n$, edges

$$\{ s^\frown 0^\frown t,\ s^\frown 1^\frown t \},\quad s \in S,\ \text{length}(s) + \text{length}(t) + 1 = n$$

(a finite "approximation" to $\mathbb{G}_S$)





$S = \{\phi, 0, 01, 101, \ldots\}$

a copy of H_3

Want a homomorphism $f: \mathcal{C} \to X$ from G_S to G

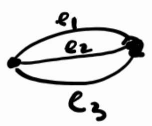
Philosophy: If we find "many" copies of H_n, then there should also be "many" copies of H_{n+1}.

Now let's formalize this.

Fix an analytic graph G on a Polish space X.

There are a Polish space E and a continuous map $\pi: E \to [X]^2$ s.t. $E(G) = \pi(E)$.

Intuition: G is a multigraph, pts of E are "names" for the edges of G



A (homomorphic) copy of a finite graph H in G is a mapping φ sending $V(H) \to X$, $E(H) \to E$ so that for all $\{u,v\} \in E(H)$, we have

① $\{\varphi(u), \varphi(v)\} \in E(G)$ and

$\Uparrow$

② $\pi(\varphi(\{u,v\})) = \{\varphi(u), \varphi(v)\}$

Intuition: A multigraph homomorphism from H to G.

Note: φ need not be injective.

If H has n vertices, m edges, can identify a copy φ with a tuple in $X^n \times E^m$. The set $\mathrm{Hom}(H,G) \subseteq X^n \times E^m$ of all copies of H in G is closed (exercise).

For a set $\mathcal{H} \subseteq \mathrm{Hom}(H,G)$ and $u \in V(H)$, $uv \in E(H)$, let

$$\left.\begin{aligned} \mathcal{H}(u) &:= \{\varphi(u) : \varphi \in \mathcal{H}\}, \\ \mathcal{H}(uv) &:= \{\varphi(uv) : \varphi \in \mathcal{H}\}. \end{aligned}\right\}$$

If $\mathcal{H}$ is Borel, these are analytic sets

↳ $x \in \mathcal{H}(u) \iff \exists \varphi \in \mathcal{H}$ s.t. $\varphi(u) = x$.

Key definition: A Borel set $\mathcal{H} \subseteq \mathrm{Hom}(H,G)$

is *tiny* if for some $u \in V(H)$, $\mathcal{H}(u)$ is G-independent,

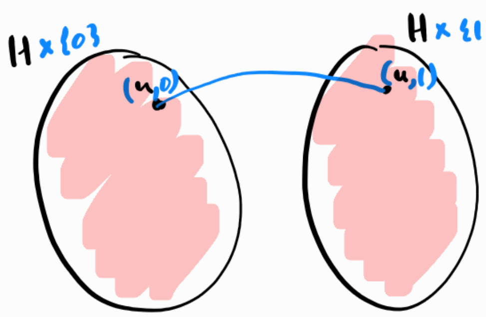
small if it can be covered by ctbly many tiny Borel sets,

large if it is not small.

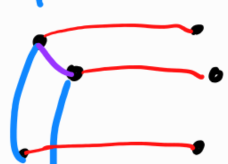
Observation: $\chi_B(G) = 2^{\aleph_0} \iff \underbrace{\mathrm{Hom}(K_1, G)}_{\cong X}$ is large.

Key operation: For $u \in V(H)$, let $H +_u H$ be the graph with vtx set $H \times \{0,1\}$, edge set

$$\{\{(v,i),(w,i)\} : vw \in E(H),\ i \in \{0,1\}\}$$
$$\cup \{\{(u,0),(u,1)\}\}.$$

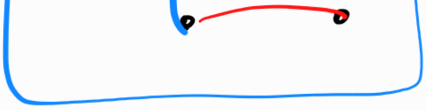


Example:



A copy $\varphi \in \mathrm{Hom}(H +_u H, G)$ of $H +_u H$ yields two copies of H, $\varphi^0, \varphi^1 \in \mathrm{Hom}(H,G)$:

$$\left[\ \varphi^i(u) := \varphi(u,i).\right.$$



$\varphi^i(uv) := \varphi((u,i),(v,i)).$

Key lemma: Let $\mathcal{H} \subseteq \mathrm{Hom}(H,G)$ be a large Borel set. Then, for any $u \in V(H)$,

$$\mathcal{H} +_u \mathcal{H} := \{ \varphi \in \mathrm{Hom}(H +_u H, G) : \varphi^0, \varphi^1 \in \mathcal{H} \}$$

is also large.