Math 223D, Lecture 17

Set-up: G an analytic graph on a Polish space X,

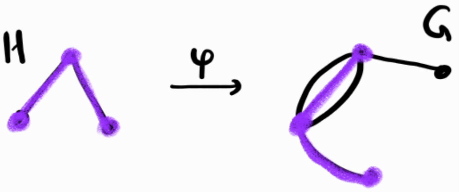
E a Polish space, $\pi: E \to [X]^2$ a continuous map s.t. $E(G) = \pi(E)$.

↳ think of (X, E, π) as a multigraph

Let H be a finite graph.

A (homomorphic) copy of H in G: a homomorphism from H to the multigraph (X, E, π), i.e.:

> a map φ sending $V(H) \to X$, $E(H) \to E$ so that for all $uv \in E(H)$:
> $$\pi(\varphi(uv)) = \{\varphi(u), \varphi(v)\}.$$



$\mathrm{Hom}(H, G)$ = space of all copies of H in G.

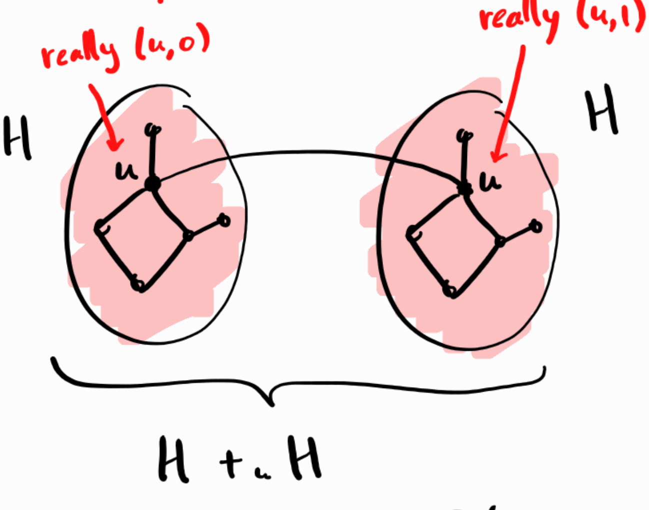
Key definition: A Borel set $\mathcal{H} \subseteq \mathrm{Hom}(H, G)$ is:

* **tiny** if for some $u \in V(H)$, the set $\mathcal{H}(u) := \{\varphi(u) : \varphi \in \mathcal{H}\}$ is G-independent,
* **small** if it is covered by ctbly many tiny Borel sets,
* **large** if it is not small.

Key operation: Given $u \in V(H)$, let $H +_u H$ be the graph with vtx set $V(H) \times \{0,1\}$, edges

$\{(v,i), (w,i)\}$ for $vw \in E(H)$, $i \in \{0,1\}$

and $\{(u,0), (u,1)\}$.

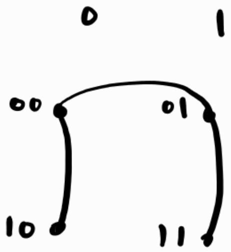


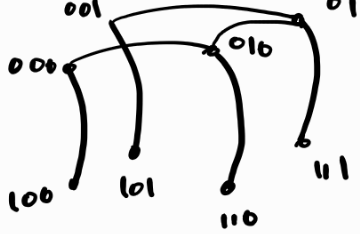
Example: $H_0 = K_1$

$H_1 = H_0 +_\varnothing H_0$

$H_3 = H_2 +_{01} H_2$

$H_2 = H_1 +_0 H_1$





.... Approximations to G_S for $S = \{\emptyset, 0, 01, \dots\}$.

Note: If φ is a copy of $H +_u H$ in G, it gives rise to two copies of H in G, φ^0 and φ^1, defined as

$$\varphi^i(v) := \varphi(v,i), \quad \varphi^i(vw) = \varphi\big((v,i),(w,i)\big).$$

Key lemma: Let H be a finite graph, $u \in V(H)$. Suppose a Borel set $\mathcal{H} \subseteq \mathrm{Hom}(H, G)$ is LARGE. Then the set

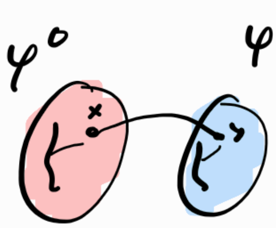
$$\mathcal{H} +_u \mathcal{H} := \{\varphi \in \mathrm{Hom}(H +_u H, G) : \varphi^0, \varphi^1 \in \mathcal{H}\}$$

is also LARGE.

Proof: Note that $\mathcal{H} +_u \mathcal{H}$ is Borel (exercise).

We first show that $\mathcal{H} +_u \mathcal{H} \neq \emptyset$.

$\mathcal{H}$ is LARGE $\Leftrightarrow$ not small $\Rightarrow$ not tiny $\Rightarrow$ $\mathcal{H}(u)$ is not G-independent, i.e., there are adjacent $x, y \in \mathcal{H}(u)$.



So there are $\varphi^0, \varphi^1 \in \mathcal{H}$ s.t. $\varphi^0(u) = x$, $\varphi^1(u) = y$.
"Combining" φ^0 and φ^1 get $\varphi \in \mathcal{H} +_u \mathcal{H}$, as desired.

Now suppose $\mathcal{H}$ is small, i.e., there are tiny Borel sets $F_n \subseteq \mathrm{Hom}(H +_u H, G)$ s.t.

$$\boxed{\mathcal{H} +_u \mathcal{H} \subseteq \bigcup_{n \in \mathbb{N}} F_n.}$$

Can pick $v_n \in V(H)$, $i_n \in \{0,1\}$ so that $\underbrace{F_n(v_n, i_n)}_{\text{analytic}}$ is G-independent.

$\Longrightarrow$ there is a Borel G-indep. set $I_n \supseteq \mathcal{F}_n(v_n, i_n)$.

↑ will prove later!

Set $\mathcal{H}_n := \{\varphi \in \mathcal{H} : \varphi(v_n) \in I_n\}$ tiny Borel subset of $\mathcal{H}$

$\Rightarrow$ $\mathcal{H}' := \mathcal{H} \setminus \bigcup_{n \in \mathbb{N}} \mathcal{H}_n$ is a LARGE Borel set

$\Rightarrow$ $\boxed{\mathcal{H}' +_n \mathcal{H}' \neq \phi.}$ But $\mathcal{H}' +_n \mathcal{H}'$ is a subset of $\mathcal{H} +_n \mathcal{H}$ disjoint from $\underline{\underline{\mathcal{F}_n}}$ for all $n \in \mathbb{N}$, so it must be empty! ↯

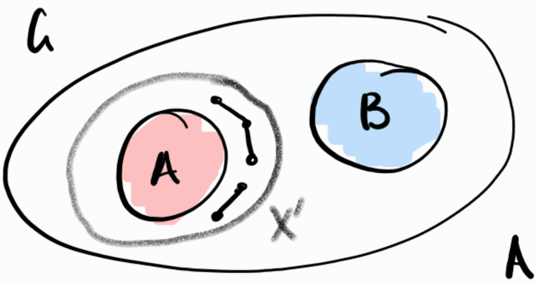
any $\varphi \in \mathcal{F}_n$ has $\varphi^{i_n}(v_n) \in I_n$, so $\varphi^{i_n} \notin \mathcal{H}'$

We have used the following:

true for any analytic graph G

Proposition: If $A \subseteq X$ is analytic, G-indep., then there is a Borel G-indep. set $I \supseteq A$.

Proof:

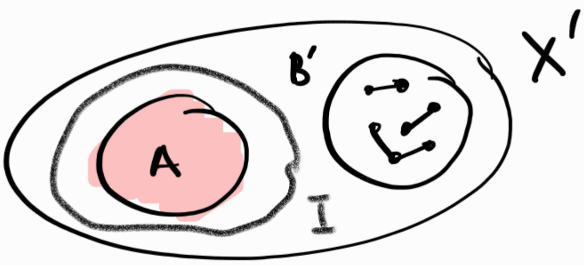


Consider $B := \{x \in X : x \text{ has a nbr in } A\}$

$$x \in B \Leftrightarrow \underbrace{\exists y \in X\,(y \in A \ \&\ xy \in E(G))}_{\text{analytic}}$$

$A \cap B = \phi$ $\Rightarrow$ By Analytic Separation, there is a Borel set X' s.t. $A \subseteq X'$, $X' \cap B = \phi$.

Now let B' be the set of all $\underline{x \in X'}$ that have $\underline{\text{a nbr in } X'}$



Again, B' is analytic, disjoint from A, so can find Borel $I \subseteq X'$ s.t. $A \subseteq I$, $I \cap B' = \phi$.

Then I is as desired.

To finish the proof, need one more lemma:

$\boxed{\text{Lemma: Fix compatible complete metrics } d \text{ on } X,\ \delta \text{ on } E.}$

Lemma: Fix compatible complete metrics d on X, δ on E. If $\mathcal{H} \subseteq \mathrm{Hom}(H, G)$ is Borel and LARGE, then for any $\varepsilon > 0$, there is a LARGE Borel set $\mathcal{H}' \subseteq \mathcal{H}$ s.t. for all $x \in V(H) \cup E(H)$, $\operatorname{diam}(\mathcal{H}'(x)) < \varepsilon$. (*)

Proof idea: Cover $\mathcal{H}$ by ctbly many Borel sets $\mathcal{H}'$ satisfying (*) ∎

Proof of G_0-dichotomy: Let $S \subset 2^*$ contain exactly one string of each length. Suppose $\chi_B(G) = 2^{\aleph_0}$. Want to show: $G_S \to_c G$.

$$S = \{s_0, s_1, s_2, \ldots\} \text{ where } \operatorname{length}(s_n) = n.$$

$H_n :=$ the graph on $\{0,1\}^n$ with edges $\{s_i{}^\frown 0^\frown t,\ s_i{}^\frown 1^\frown t\}$, $i < n$, $i + 1 + \operatorname{length}(t) = n$.

Then $\boxed{H_{n+1} = H_n +_{s_n} H_n.}$

$H_0 \cong K_1 \Rightarrow$ As $\chi_B(G) = 2^{\aleph_0}$, $\mathcal{H}_0 := \mathrm{Hom}(H_0, G)$ is LARGE

Now define $\mathcal{H}_{n+1} \subseteq \mathrm{Hom}(H_{n+1}, G)$ so that:

- $\mathcal{H}_{n+1}$ is Borel and LARGE
- $\mathcal{H}_{n+1} \subseteq \mathcal{H}_n +_{s_n} \mathcal{H}_n$
- for all $x \in V(H_{n+1}) \cup E(H_{n+1})$, $\operatorname{diam}(\mathcal{H}_{n+1}(x)) < \frac{1}{2^n}$.

this implies that for all $u, v \in \{0,1\}^n$, $i \in \{0,1\}$,

$$\mathcal{H}_{n+1}(u^\frown i) \subseteq \mathcal{H}_n(u), \qquad \mathcal{H}_{n+1}(\{u^\frown i, v^\frown i\}) \subseteq \mathcal{H}_n(\{u, v\}).$$

$\Rightarrow$ Can let, for any $x = x_0 x_1 x_2 \ldots \in \mathcal{C}$,

$f(x) :=$ the unique point in $\bigcap \overline{\mathcal{H}_n(x_0 \ldots x_{n})}$

$f(x) :=$ the unique pt in $\bigcap_{n\in\mathbb{N}} \mathcal{H}_n(x_0 x_1 \dots x_{n-1})$

$f : \mathcal{C} \to X$ is continuous (exercise) and a homomorphism from

G_S to G: Take $\{x,y\} \in E(G_S)$. Say

$$\begin{cases} x = s_i{}^\frown 0^\frown t \\ y = s_i{}^\frown 1^\frown t \end{cases}$$

an edge in H_n

Let e be the unique pt in $\bigcap_{n \geq i+1} \mathcal{H}_n\left(\overline{\{x_0 x_1 \dots x_{n-1}, y_0 y_1 \dots y_{n-1}\}}\right)$

Since π is continuous, $\pi(e) = \{f(x), f(y)\}$,

so $\{f(x), f(y)\} \in E(G)$, as desired.