Math 223D, Lecture 18

Proof of G_0-dichotomy, final part

Recall the set-up:

X, E Polish, $\pi : E \to [X]^2$ continuous, $G = (X, \pi(E))$ an analytic graph.

For finite graphs H, look at (homomorphic) copies of H in G

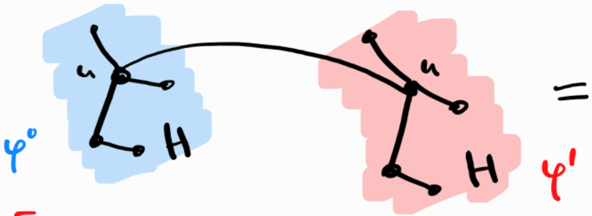
$$\varphi : V(H) \to X,\ E(H) \to E \quad + \text{ compatibility property}$$

$\mathrm{Hom}(H, G) \subseteq X^n \times E^m$ closed

$n = |V(H)|$, $m = |E(H)|$

A Borel set $\mathcal{H} \subseteq \mathrm{Hom}(H, G)$ is

- **tiny** if for some $v \in V(H)$, $\mathcal{H}(v) := \{\varphi(v) : \varphi \in \mathcal{H}\}$ is G-indep.,
- **small** if it is covered by ctbly many tiny Borel sets,
- **LARGE** if it isn't small.



$=: H +_u H$

Key lemma: If $\mathcal{H} \subseteq \mathrm{Hom}(H, G)$ is a LARGE Borel set, then

$$\mathcal{H} +_u \mathcal{H} := \{\varphi \in \mathrm{Hom}(H +_u H, G) : \varphi^0, \varphi^1 \in \mathcal{H}\}$$

is also LARGE.

To finish the proof, need one more lemma:

Lemma: Fix compatible complete metrics d on X, δ on E.
If $\mathcal{H} \subseteq \mathrm{Hom}(H, G)$ is Borel and LARGE, then for any $\varepsilon > 0$, there is a LARGE Borel set $\mathcal{H}' \subseteq \mathcal{H}$ s.t. for all $x \in V(H) \cup E(H)$,

$$\mathrm{diam}(\mathcal{H}'(x)) < \varepsilon. \quad (*)$$

Proof idea: Cover $\mathcal{H}$ by ctbly many Borel sets $\mathcal{H}'$ satisfying $(*)$ ∎

Proof of G_0-dichotomy: Let $S \subset 2^*$ contain exactly one string of each length. Suppose $\chi_B(G) = 2^{\aleph_0}$. Want to show: $G_S \to_c G$.

$S = \{s_0, s_1, s_2, \ldots\}$ where $\mathrm{length}(s_n) = n$.

$H_n :=$ the graph on $\{0,1\}^n$ with edges $\{s_i{}^\frown 0^\frown t,\ s_i{}^\frown 1^\frown t\}$, $i < n$, $i + 1 + \mathrm{length}(t) = n$.

Then $\boxed{H_{n+1} = H_n +_{s_n} H_n.}$

$H_0 \cong K_1 \Rightarrow$ As $\chi_B(G) = 2^{\aleph_0}$, $\mathcal{H}_0 := \mathrm{Hom}(H_0, G)$ is LARGE

Now define $\mathcal{H}_{n+1} \subseteq \mathrm{Hom}(H_{n+1}, G)$ so that:

- $\mathcal{H}_{n+1}$ is Borel and LARGE
- $\mathcal{H}_{n+1} \subseteq \mathcal{H}_n +_{s_n} \mathcal{H}_n$
- for all $x \in V(H_{n+1}) \cup E(H_{n+1})$, $\mathrm{diam}(\mathcal{H}_{n+1}(x)) < \frac{1}{2^n}$.

this implies that for all $u, v \in \{0,1\}^n$, $i \in \{0,1\}$,

$\mathcal{H}_{n+1}(u^\frown i) \subseteq \mathcal{H}_n(u)$, $\quad \mathcal{H}_{n+1}(\{u^\frown i, v^\frown i\}) \subseteq \mathcal{H}_n(\{u,v\})$.

$\Rightarrow$ Can let, for any $x = x_0 x_1 x_2 \ldots \in \mathcal{C}$,

$$f(x) := \text{the unique pt in } \bigcap_{n \in \mathbb{N}} \overline{\mathcal{H}_n(x_0 x_1 \ldots x_{n-1})}$$

$f : \mathcal{C} \to X$ is continuous (exercise) and a homomorphism from G_S to G: Take $\{x,y\} \in E(G_S)$. Say

$$\begin{cases} x = s_i{}^\frown 0^\frown t \\ y = s_i{}^\frown 1^\frown t \end{cases}$$

an edge in H_n

Let $e \in E$ be the unique pt in $\bigcap_{n \geq i+1} \mathcal{H}_n\left(\{x_0x_1 \ldots x_{n-1}, y_0y_1 \ldots y_{n-1}\}\right)$

Since π is continuous, $\pi(e) = \{f(x), f(y)\}$,

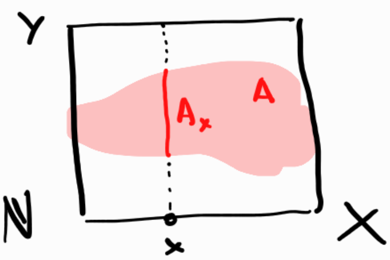
so $\{f(x), f(y)\} \in E(\mathcal{G})$, as desired.

Applications of $\mathbb{G}_0$-dichotomy

Thm (Luzin-Novikov) Let X, Y be st. Borel, $A \subseteq X \times Y$ a Borel set.

For $x \in X$, let $A_x := \{y \in Y : (x,y) \in A\}$.

If A_x is ctbl for all $x \in X$, then there exist partial Borel functions $f_n : X \dashrightarrow Y$, $n \in \mathbb{N}$



dom$(f_n) \subseteq X$ Borel

such that $(x,y) \in A \iff y = f_n(x)$ for some $n \in \mathbb{N}$.

(i.e., $A_x = \{f_n(x) : x \in \mathrm{dom}(f_n)\}$).

Proof: Next time

Extremely usefull! For example:

Corollary: If $\mathcal{G}$ is a loc. ctbl Borel graph, then the set $\{v \in V(\mathcal{G}) : \deg_{\mathcal{G}}(x) = 0\}$ is Borel.

In general, this set is co-analytic.

Pf: Apply LN to $A := \{(u,v) \in V(\mathcal{G}) \times V(\mathcal{G}) : \{u,v\} \in E(\mathcal{G})\}$.

$\rightsquigarrow f_n : V(\mathcal{G}) \dashrightarrow V(\mathcal{G})$, $n \in \mathbb{N}$ s.t.

$$\{u,v\} \in E(G) \iff v = f_n(u) \text{ for some } n \in \mathbb{N}.$$

Then $\{v \in V(G) : \deg_G(v) = 0\} = V(G) \setminus \bigcup_{n \in \mathbb{N}} \operatorname{dom}(f_n)$.