Math 223D, Lecture 19 | Next lecture (May 14): ON ZOOM!

Applications of $\mathbb{G}_0$-dichotomy

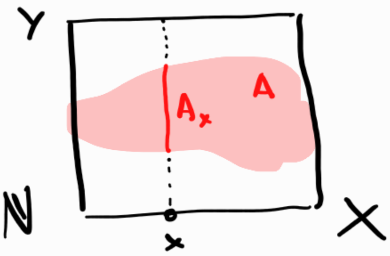
Thm (Luzin–Novikov) Let X, Y be st. Borel, $A \subseteq X \times Y$ a Borel set.

For $x \in X$, let $A_x := \{y \in Y : (x,y) \in A\}$.

If A_x is ctbl for all $x \in X$, then there exist partial Borel functions $f_n : X \dashrightarrow Y$, $n \in \mathbb{N}$

(partial: $\mathrm{dom}(f_n) \subseteq X$ Borel)

such that $(x,y) \in A \iff y = f_n(x)$ for some $n \in \mathbb{N}$.



(i.e., $A_x = \{f_n(x) : x \in \mathrm{dom}(f_n)\}$).

Proof (Miller): Assume X, Y are Polish.

For $x \in X$, let G_x be the graph on Y with

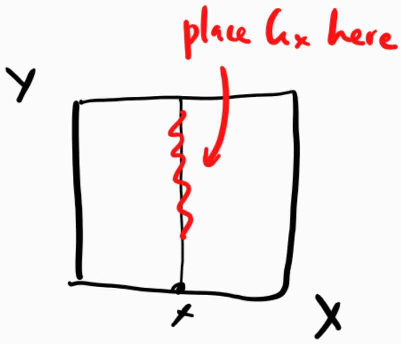
$$\{y, y'\} \in E(G_x) \iff y \neq y' \text{ and } y, y' \in A_x.$$

Note: $\boxed{\chi_B(G_x) \leq \aleph_0}$: The following map is a ctbl Borel coloring of G_x:

(because A_x is ctbl)

$$y \longmapsto \begin{cases} y & \text{if } y \in A_x \\ 0 & \text{if } y \notin A_x \end{cases}$$

(0: arbitrary)



Now let G be the graph on $X \times Y$ with

$$\{(x,y), (x',y')\} \in E(G) \iff x = x' \text{ and } \{y, y'\} \in G_x$$

We claim that $\boxed{\chi_B(G) \leq \aleph_0}$:

If not, there is a continuous hom. $h : \mathcal{C} \to X \times Y$ from G_S to G, where $S \subset 2^*$ is dense and contains one string of each length.

Write $h = (h_1, h_2)$ where $h_1 : \mathcal{C} \to X$, $h_2 : \mathcal{C} \to Y$.

Note: h_1 is constant on components of G_S (because h is a hom.)

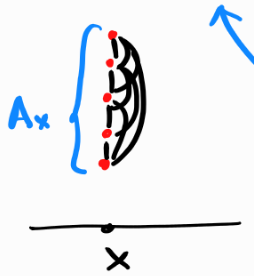
But the components of G_S (same as comps of $\mathbb{H}$) are dense in $\mathcal{C}$

Since h_1 is continuous $\Longrightarrow$ $\boxed{h_1 \text{ is constant.}}$

Let $x \in X$ be the fixed value of h_1. Then h_2 is a homomorphism from G_S to G_x. But that's impossible as $\chi_B(G_x) \leq \aleph_0$.

Let $c : X \times Y \to \mathbb{N}$ be a Borel proper coloring of G.

Define $f_n(x) :=$ the unique y (if it exists) s.t. $y \in A_x$ and $c(x,y) = n$ (because c is a proper coloring)



as desired! ☺ ∎

[**Corollary**: X, Y st. Borel, $f : X \to Y$ ctbl-to-one Borel map.
Then $f(X)$ is a Borel subset of Y.]

(in general $f(X)$ is analytic...)

Pf: Apply LN to the set $A := \{(y,x) \in Y \times X : y = f(x)\}$.

$\rightsquigarrow$ get Borel partial maps $f_n : Y \dashrightarrow X$, $n \in \mathbb{N}$ s.t.

$$A_y = \{x \in X : f(x) = y\} = f^{-1}(y) = \underbrace{\{ f_n(y) : n \in \mathbb{N} \text{ s.t. } y \in \mathrm{dom}(f_n)\}}.$$

Hence, $f(X) = \bigcup_{n \in \mathbb{N}} \mathrm{dom}(f_n)$. ← Borel ☺ ∎

Very useful for working with locally ctbl graphs

Defn: Given a graph G, let $\sim_G$ be the equivalence relation on $V(G)$ where $u \sim_G v \iff u$ and v are in the same component of G (i.e., G contains a uv-path).

Note: If G is a Borel graph, the relation $\sim_G$ is analytic:

$$u \sim_G v \iff \exists k \in \mathbb{N} \underbrace{\exists u_0, u_1, \ldots, u_k}_{☹} \in V(G) \text{ s.t. } \begin{cases} u_0 = u, \\ u_0u_1, \ldots, u_{k-1}u_k \in E(G), \\ u_k = v \end{cases}$$

Prop: If G is a locally ctbl Borel graph, then the relation $\sim_G$ is Borel.

Pf. Apply LN to get Borel maps $f_n : V(G) \dashrightarrow V(G)$, $n \in \mathbb{N}$ s.t. for all $v \in V(G)$, $\underbrace{N_G(v)}_{\text{nbhd of } v \text{ in } G} = \{ f_n(v) : n \in \mathbb{N}, v \in \mathrm{dom}(f_n) \}$.

Now: $u \sim_G v \iff \exists k \in \mathbb{N} \underbrace{\exists n_1, \ldots, n_k \in \mathbb{N}}_{☺}$ s.t.

$$u \in \mathrm{dom}(f_1),\ f_1(u) \in \mathrm{dom}(f_2), \ldots\ldots, f_{k-1}(\cdots(f_1(u))\cdots) \in \mathrm{dom}(f_k),$$

$$f_k(f_{k-1}(\cdots(u)\cdots)) = v. \quad \blacksquare$$

Exercise: If G is a loc. ctbl Borel graph, then the map $\deg_G : V(G) \to \mathbb{N} \cup \{\infty\}$ is Borel.

Thm: If G is a locally finite Borel graph, then $\boxed{\chi_B(G) \leq \aleph_0}$

Pf. Fix a ctbl basis $(U_i)_{i \in \mathbb{N}}$ for a compatible Polish top. on $V(G)$.

Define: $c(u) := \min\{ i \in \mathbb{N} : u \in U_i \text{ and no nbr of } u \text{ is in } U_i \}$.

Well-defined:  U_i $\}$ finitely many   Proper ✓

exercise

<u>Borel?</u> ⇝ Apply Luzin-Novikov! Enough to check that for all $i \in \mathbb{N}$, the set $N_G(U_i) := \{u : u \text{ has a nbr in } U_i\}$ is Borel. Let $f_n : V(G) \dashrightarrow V(G)$ be Borel maps s.t.

$\{u,v\} \in E(G) \iff v = f_n(u)$ for some $n \in \mathbb{N}$. Then

$$N_G(U_i) = \bigcup_{n \in \mathbb{N}} f_n^{-1}(U_i) \quad \text{Borel} \ \smile \qquad \blacksquare$$