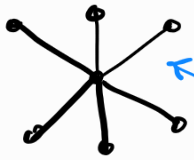
Math 223D, Lecture 20 ☺ More on locally ctbl graphs

Proper edge-coloring of G: $c : E(G) \to Y$ s.t. $c(e) \neq c(e')$ when $|e \cap e'| = 1$



all these edges have different colors

Chromatic index of G: $\chi'(G) = \min |Y|$ s.t. there exists a proper edge-coloring $c : E(G) \to Y$.

Equivalent formulation: The line graph of G,

$L(G)$: vertices $= E(G)$

edges $= \{ \{e, e'\} \in [E(G)]^2 : e \cap e' \neq \varnothing \}$.

Then $\chi'(G) = \chi(L(G))$.

Define $\chi'_B(G)$ analogously (when G is Borel).

Thm (Feldman-Moore / Kechris-Solecki-Todorcevic):
Let G be a Borel graph. Then
$\chi'_B(G) \leq \aleph_0 \iff \chi'(G) \leq \aleph_0 \iff G$ is locally ctbl.

Pf: ($\Rightarrow$) Clear (edges incident to a vtx have different colors)

$\Leftarrow$: Suppose G is locally ctbl. [Want: a ctbl Borel proper edge-coloring]

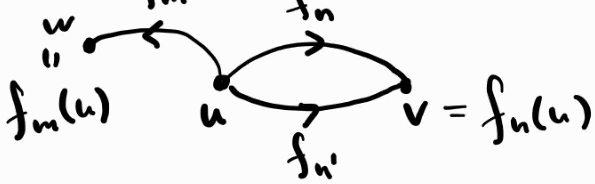
Apply Luzin-Novikov:

$\leadsto$ Borel maps $f_n : V(G) \dashrightarrow V(G)$, $n \in \mathbb{N}$ s.t.

$\{u, v\} \in E(G) \iff v = f_n(u)$ for some $n \in \mathbb{N}$.

Idea: If $v = f_n(u)$, color the edge $\{u, v\}$ with the color n.

Can have $f_n(u) = f_m(u)$



Issue #1: Can ...
for some $n \neq n'$

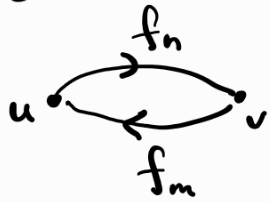
Fix: Define $\ell(u,v) := \min\{n \in \mathbb{N} : v = f_n(u)\}$.

Now, if $\{u,v\}, \{u,w\} \in E(G)$ are two distinct edges,

then $\ell(u,v) \neq \ell(u,w)$.

Issue #2: ℓ is a function of the ordered pair (u,v),

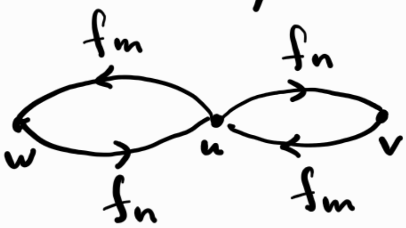
i.e., $\ell(u,v)$ and $\ell(v,u)$ may be different.



Fix: Define $c : E(G) \to [\mathbb{N}]^1 \cup [\mathbb{N}]^2$ by

$$c(\{u,v\}) = \{\ell(u,v), \ell(v,u)\}.$$ Borel (check!)

Issue #3: This may not be a proper coloring!



$c = \{n, m\}$

Fix: Let H be the spanning subgraph of $L(G)$

$(V(H) = E(G))$ with edge set

$$E(H) = \{\{e,e'\} : |e \cap e'| = 1 \text{ and } c(e) = c(e')\}.$$

Claim: H is locally finite; in fact, its max. deg. is ≤ 2.

↳ Why? Suppose we have distinct edges $\{u,v\}, \{u,w\}$

s.t. $c(\{u,v\}) = c(\{u,w\}) = \{n,m\}$, where

$f_n(u) = v$, $f_m(v) = u$. Since $w \neq v = f_n(u)$, we must

have $w = f_m(u)$, so given u, v, there is ≤ 1 such w.

have $w = f_m(u)$, so given u, v, there is ≤ 1 ...

Similarly, have ≤ 1 edge $\{v, x\}$ s.t. $c(\{v,x\}) = c(\{u,v\})$. ☒

$1+1=2$.

H is loc. finite $\Rightarrow \chi_B(H) \leq \aleph_0$.

Let $c' : E(G) \to \mathbb{N}$ be a Borel proper coloring of H.

Now, $(c, c') : E(G) \to [\mathbb{N}]^{\leq 2} \times \mathbb{N}$ is a Borel proper edge-coloring of G. ☺ ☒

Nice application:

Thm (Injective G_0-dichotomy): Let $S \subset 2^*$ be a set containing one string of each finite length. If G is a locally ctbl Borel graph with $\chi_B(G) = 2^{\aleph_0}$, then there exists an injective continuous hom. from G_S to G.

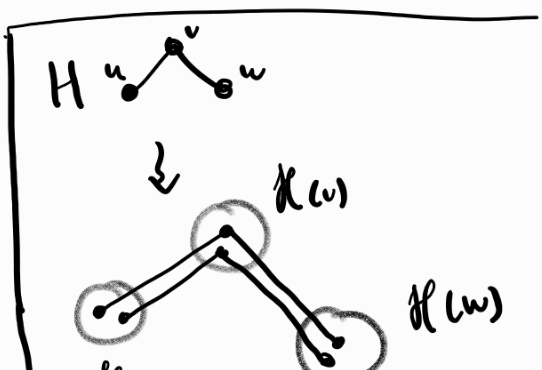
"G has a subgraph Borel-isomorphic to G_S"

Rmk (Lecomte): For non-locally ctbl G, this may fail.

Proof sketch (details: homework!)

A set $\mathcal{H} \subseteq \mathrm{Hom}(H, G)$ of copies of a finite graph H in G is **injective** if for all $\varphi \in \mathcal{H}$, φ is injective on $V(H)$ (i.e., φ is an injective hom $H \to G$)

strongly injective: if for any $u, v \in V(H)$, $u \neq v$, we have $\underbrace{\mathcal{H}(u) \cap \mathcal{H}(v)}_{\subseteq V(G)} = \varnothing$



$\mathcal{H}(u) = \{\varphi(u) : \varphi \in \mathcal{H}\}$

$\mathcal{H}(u)$

Fact #1: If $\mathcal{H} \subseteq \mathrm{Hom}(H, G)$ is Borel, large, and injective, then it has a large Borel subset $\mathcal{H}' \subseteq \mathcal{H}$ that is strongly injective.

Pf idea: Cover $\mathcal{H}$ by ctbly many strongly injective Borel sets.

Fix a Borel proper edge-coloring $c : E(G) \to \mathbb{N}$ (Feldman-Moore)

A set $\mathcal{H} \subseteq \mathrm{Hom}(H, G)$ is **consistent** if for each $uv \in E(H)$, the color $c(\underbrace{\{\varphi(u), \varphi(v)\}}_{\in E(G)})$ is the same for all $\varphi \in \mathcal{H}$.

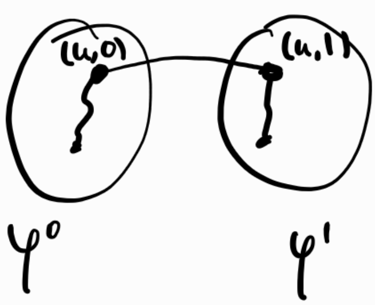
Fact #2: If $\mathcal{H} \subseteq \mathrm{Hom}(H, G)$ is Borel and large, then it has a large Borel subset $\mathcal{H}' \subseteq \mathcal{H}$ that is consistent.

Pf idea: Cover $\mathcal{H}$ by ctbly many Borel consistent sets.

Finally:

KEY Fact #3: If $\mathcal{H} \subseteq \mathrm{Hom}(H, G)$ is consistent, strongly injective, and $u \in V(H)$, then
$$\mathcal{H} +_u \mathcal{H} \subseteq \mathrm{Hom}(H +_u H, G) \text{ is injective.}$$

Pf: Take $\varphi \in \mathcal{H} +_u \mathcal{H}$ so $\varphi^0, \varphi^1 \in \mathcal{H}$



Need to show $\varphi : V(H) \times \{0,1\} \to V(G)$ is injective.

Suppose not: $\underbrace{\varphi(v, i)}_{\varphi^i(v)} = \underbrace{\varphi(v', i')}_{\varphi^{i'}(v')}$ for $(v, i) \neq (v', i')$.

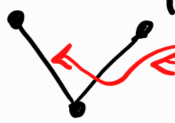
Note: $\mathcal{H}$ is strongly injective, so if $\varphi^i(v) = \varphi^{i'}(v')$, then $\boxed{v = v'}$. So may assume $\boxed{\varphi^0(v) = \varphi'(v)}$.

Pick such $v \in V(H)$ at min distance from u.

Note: $v \neq u$ because $\varphi^0(u)$ and $\varphi'(u)$ are adjacent, hence distinct.

But now consider the last vtx w on a shortest uv-path:

$\varphi^0(w) \neq \varphi'(w)$ by the choice of v.



$\varphi^0(v) = \varphi'(v)$

these edges have the same color!

That's a contradiction! $\boxtimes$

With these facts, can run the same construction as in the proof of G_0-dichotomy but additionaly requiring $\mathcal{H}_n$ to be $\underbrace{\text{strongly injective}}$, and consistent.

$\hookrightarrow$ ensures the final hom. is injective. $\boxtimes$

$$f(x) = \text{the unique pt in } \bigcap_{n \in \mathbb{N}} \overline{\mathcal{H}_n(x_0 x_1 \cdots x_{n-1})}$$