Next time (Monday, May 19): Homework discussion (bring questions/comments!)

Borel graphs of bounded degree

Very useful fact:

e.g., if G is loc. finite

Prop: Let G be a locally ctbl Borel graph with $\chi_B(G) \leq \aleph_0$.
Then G has a Borel maximal independent set $I \subseteq V(G)$.

no vtx can be added, i.e., each $x \in V(G) \setminus I$ has a nbr in I.

Pf: Let $c : V(G) \to \mathbb{N}$ be a Borel proper coloring.
Let $I_n := c^{-1}(n)$, so $V(G) = I_0 \cup I_1 \cup \ldots$ is a partition into Borel indep. sets. Recursively, for $n = 0, 1, 2, \ldots$, define

$$I_n' := \{x \in I_n : x \text{ has no nbr in } I_0' \cup \ldots \cup I_{n-1}'\}$$

$$= I_n \setminus \underbrace{N_G}_{\text{nbhd}}(I_0' \cup \ldots \cup I_{n-1}') \to \text{Borel by Luzin-Novikov}$$

Then $I := \bigcup_{n \in \mathbb{N}} I_n'$ is as desired. ∎

Corollary (Kechris-Solecki-Todorcevic): If G is a Borel graph of max. degree $d \in \mathbb{N}$, then $\chi_B(G) \leq d+1$.

Pf: Induction on d. For $d = 0 \rightsquigarrow$ clear.
If $d \geq 1$: Pick a Borel maximal ind. set $I \subseteq V(G)$.
Then $G - I$ is a Borel graph of max. deg. $\leq d-1$

$\implies \chi_B(G - I) \leq (d-1)+1 = d$ by the inductive hypothesis.

Extend a d-coloring of $G - I$ to a $(d+1)$-coloring of G by giving all vertices in I the same (new) color. ∎

In general, $d+1$ is best possible

Example 1: If G has a component $\cong K_{d+1}$, then

$$\chi_B(G) = \chi(G) = d+1.$$

Example 2: If $d=2$ and G has a comp. $\cong$ an odd cycle,

then $\chi_B(G) = \chi(G) = 3 = 2+1$.

Note: **Thm** (Brooks) If G is a graph of max. deg. $d \in \mathbb{N}$, then $\boxed{\chi(G) \leq d}$ unless G has a comp. $\cong K_{d+1}$ or $d=2$ and G has a comp. $\cong$ an odd cycle.

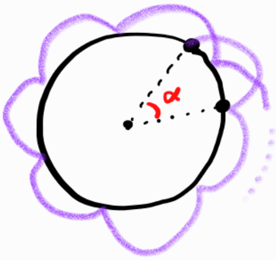
However:

circle

Example 3: For $\alpha \in \mathbb{R}$, let $T_\alpha : S^1 \to S^1$ be rotation by α.

Claim: If $\frac{\alpha}{\pi} \notin \mathbb{Q}$, then G_{T_α} is an acyclic Borel graph of max deg 2 and $\chi_B = 3$

Pf:



Max deg 2 ⇝ clear

Acyclic ⇝ clear as $\frac{\alpha}{\pi} \notin \mathbb{Q}$.

$\chi_B \leq 2+1 = 3$ ⇝ clear

Need to show: $\chi_B \geq 3$.

Lebesgue measure

Actually, χ_μ and $\chi_{BM} \geq 3$ as well.

usual topology

Say, suppose $S^1 = A_0 \sqcup A_1$ is a partition into two measurable indep. sets. Then

$$T_\alpha(A_0) = A_1, \quad \text{so} \quad \mu(A_0) = \mu(A_1) = \frac{1}{2}.$$

Take any $\varepsilon > 0$.

99% lemma: Find non-$\varnothing$ open $U_0 \supseteq A_0$ s.t. $\dfrac{\mu(A_0)}{\mu(U_0)} > 1-\varepsilon$.

... of ctbly many open intervals

Open sets in S^1 are disjoint unions of [illegible]

$$U_0 = \bigsqcup_{k\in\mathbb{N}} I_{0k} \implies \text{For some } k, \text{ have } \frac{\mu(A_0 \cap I_{0k})}{\mu(\underbrace{I_{0k}}_{I_0})} > 1-\varepsilon$$

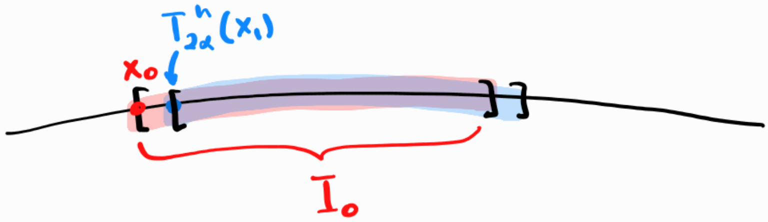
Then we also have $$\frac{\mu(A_1 \cap T_\alpha(I_{0k}))}{\mu(\underbrace{T_\alpha(I_{0k})}_{I_1})} > 1-\varepsilon.$$

Upshot: Have open intervals I_0, I_1 of the same length s.t.

$$\boxed{\frac{\mu(A_i \cap I_i)}{\mu(I_i)} > 1-\varepsilon.}$$

Fact (Dirichlet): If $\frac{\beta}{\pi} \notin \mathbb{Q}$, then for all $x \in S^1$, the set

$$\{T_\beta^n(x) : n \in \mathbb{N}\} \text{ is dense.} \quad \text{(exercise!)}$$



Let x_i be the starting pt of I_i. By [the Fact], for any $\delta > 0$, there is $n \in \mathbb{N}$

s.t. $\underbrace{T_{2\alpha}^n(x_1)}_{= T_\alpha^{2n}(x_1)}$ is at distance $< \delta$ from x_0

$\implies > 1-\varepsilon$ fraction of $T_\alpha^{2n}(I_1)$ is in A_1.

But then

$$0 = \mu(A_0 \cap A_1) \geq \mu(I_0 \cap A_0) - \mu(I_0 \setminus T_\alpha^{2n}(I_1)) - \mu(T_\alpha^{2n}(I_1) \setminus A_1)$$

$$\geq (1-\varepsilon)\mu(I_0) - \delta - \varepsilon\,\mu(I_0)$$

$$\geq (1-2\varepsilon)\mu(I_0) - \delta > 0 \quad \text{if } \varepsilon < \tfrac{1}{2} \text{ and } \delta \text{ is small enough } \lightning$$

Proof for $\chi_{Bor}(G_T)$: exercise!

Proof for $\chi_{BM}($[illegible]$)$

Survey of further results:

Max. value of $\chi_B, \chi_\mu, \ldots$ for a Borel forest of max. deg. d:

d	0	1	2	3	4	5	...	$d \to \infty$
χ	1	2	2	2	2	2	....	2
χ_B	1	2	3	4	5	6	...	$d+1$
χ_μ	1	2	3	3	3 or 4	3,4,5?	...	$\sim \frac{d}{\log d}$
χ_{BM}	1	2	3	3	3	3	...	3