MATH 223D, Lecture 22

Next Lecture (Wed, May 21): Zoom

The one after that (Fri, May 23): Asynchronous

Review on Borel graphs of bdd degree:

Let G be a Borel graph of max. deg. $d \in \mathbb{N}$

We know: Thm (KST): $\chi_B(G) \leq d+1$.

Thm (Marks '16): For any $d \in \mathbb{N}$, there exists a Borel forest of max. deg d with $\chi_B = d+1$.

Thm (Conley-Marks-Tucker-Drob '16): For $d \geq 3$, Brooks' thm holds measurably / Baire-measurably: If G has no K_{d+1}, then $\chi_\mu(G), \chi_{BM}(G) \leq d$ (for any prob. measure μ / comp. Polish top. on $V(G)$).

+ a structural analysis of graphs with $\chi_B = d+1$

We'll prove these!

Almost sharp! Fails for $a \geq \sqrt{d-\frac{3}{4}} - \frac{1}{2}$

Exact threshold? OPEN

Thm (AB '23): If $a \leq \sqrt{d+\frac{1}{4}} - \frac{5}{2}$ $(\approx \sqrt{d})$, then

$$\chi_\mu(G) \leq d-a \iff \chi_{BM}(G) \leq d-a \iff \chi(G) \leq d-a.$$

(Builds on $\underbrace{\text{Molloy-Reed '14}}_{\text{graph theory}}$, $\underbrace{\text{Bamas-Esperet '19}}_{\text{computer science}}$)

Sharp (Weilacher '22)

Thm (Conley-Miller '16): If G is a locally finite Borel graph with $\chi(G) < \infty$, then $\boxed{\chi_{BM}(G) \leq 2\chi(G) - 1.}$

starting pt of an active area: large-scale geometry of Borel graphs

Thm (Rahman-Virág '17): For any $d \in \mathbb{N}$, there exist a Borel forest G of max. deg d and a prob. measure μ on $V(G)$ s.t.

any μ-meas. G-indep. set $I \subseteq V(G)$ satisfies

$$\mu(I) \le (1+o(1)) \frac{\log d}{d}.$$

Hence, $\chi_\mu(G) \ge (1-o(1)) \dfrac{d}{\log d}$.

Buids on Bollobás, McKay (80s), Frieze-Łuczak '92 (random graphs).

Lyons – Nazarov '11 (connection to χ_μ: proved the bound up to a factor of 2).

Gamarnik - Sudan '14 (better constant)

<u>Thm</u> (AB'23): If G has no $\triangle$, $\square$:

$$\chi_\mu(G) \le (1+o(1)) \frac{d}{\log d}.$$

<u>Builds on</u>: Kim '95 (graph theory), Chung - Pettie - Su '17 (CS)

<u>Also</u>: If G is triangle-free,

$$\chi_\mu(G) \le (4+o(1)) \frac{d}{\log d}$$

<u>OPEN</u>: Can 4 be replaced by 1? Or any constant < 4?