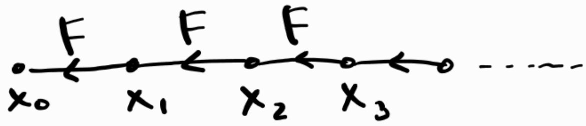
Math 223D, Lecture 23 | **Structure of graphs with $\chi_B = d+1$**

Key tool: one-ended subforests (Conley, Marks, Tucker-Drob)

↳ Along the way: useful BIG ideas

Defn: Let $F : X \dashrightarrow X$ be a partial function.

A sequence $x_0, x_1, \ldots \in X$ is **F-descending** if $F(x_{n+1}) = x_n \ \forall n$

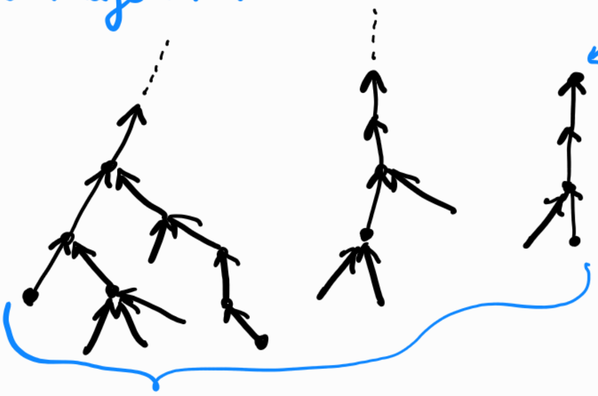


F is **one-ended** if there is no infinite F-descending sequence.

not in the image of F

"can only go forward"

G_F



$\notin \mathrm{dom}(F)$

$\notin \mathrm{im}(F)$

Note: If F is one-ended, G_F is a forest

Defn: G a graph, $A \subseteq V(G)$.

An **A-one-ended subforest** of G is a one-ended function $F : A^c \to V(G)$ s.t. $\{x, F(x)\} \in E(G) \ \forall x \in A^c$

(i.e., $G_F \subseteq G$)

Prop: G Borel graph, max. deg $d \in \mathbb{N}$. $A \subseteq V(G)$ Borel.
Suppose there is a Borel A-one-ended subforest F of G.
Then $\boxed{\chi_B(G - A) \le d}$

In particular, if there is a Borel ϕ-one-ended subforest, then $\chi_B(G) \leq d$.

Proof: $F: A^c \to V(G)$, A-one-ended.

Idea: Greedy coloring so that x is colored before $F(x)$

Define $\mathrm{rank}_F(x) :=$ sup length of an F-desc. sequence starting with x.

BIG idea #1 (Kőnig's lemma):

$\mathrm{rank}_F(x)$ is finite for all x.

Why? Suppose not: $\mathrm{rank}_F(x) = \infty$.

y must be a nbr of x

$$\infty = \mathrm{rank}_F(x) = 1 + \sup\{\mathrm{rank}_F(y) : F(y) = x\}$$

G is locally finite, so

$$= 1 + \max\{\mathrm{rank}_F(y) : F(y) = x\},$$

hence there is y s.t. $F(y) = x$ and $\mathrm{rank}_F(y) = \infty$.

Iterate this to get an infinite sequence $x_0 = x, x_1 = y, x_2, x_3, \ldots$

s.t. $\mathrm{rank}_F(x_n) = \infty$ and $F(x_{n+1}) = x_n$ for all $n \in \mathbb{N}$.

But then F is not one-ended! $\lightning$

Note: $\mathrm{rank}_F(F(x)) > \mathrm{rank}_F(x)$ for all x.

Let $V_n := \{x : \mathrm{rank}_F(x) = n\} \leftarrow$ this is a Borel set (by Luzin-Novikov)

Key property: every vtx in $V_n \cap A^c$ has $\leq d-1$ nbrs in $V_0 \cup V_1 \cup \ldots \cup V_n$.

<u>Refine:</u> There exists a partition $V(G) = \bigcup_{n \in \mathbb{N}} W_n$ into Borel G-indep. sets s.t. every vtx in $W_n \cap A^c$ has $\leq d-1$ nbrs in $W_0 \cup W_1 \cup \dots \cup W_{n-1}$.

<u>Why?</u> $\chi_B(G) \leq d+1$, so we can fix a partition $V(G) = I_0 \cup I_1 \cup \dots \cup I_d$ into Borel indep. sets.
Define $W_{(d+1)i+r} := V_i \cap I_r$ where $i \in \mathbb{N}$, $0 \leq r \leq d$
(partition each V_i into $d+1$ indep. sets).

Finally, define a coloring $c : A^c \to \{0, 1, \dots, d-1\}$ by letting, for each $x \in A^c \cap W_n$:

$$c(x) := \min\{k : c(y) \neq k \text{ for all nbrs } y \text{ of } x \text{ in } W_0 \cup \dots \cup W_{n-1}\}$$

(recursion on n) $\blacksquare$

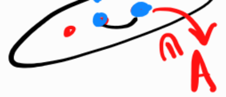
Proposition: G a locally finite Borel graph, $A \subseteq V(G)$ Borel set. Suppose that A meets every connected component of G. Then G has a Borel A-one-ended subforest.

Before proving this, an <u>application:</u>

<u>Corollary:</u> Suppose G is a Borel graph of max. deg $d \in \mathbb{N}$. If every component of G has a vtx of deg $< d$, then $\chi_B(G) \leq d$.

<u>Proof:</u> Let $A := \{x \in V(G) : \deg(x) < d\}$.

Take a <u>Borel maximal indep. set</u> $I \subseteq A$. $\in I$ $\notin I$

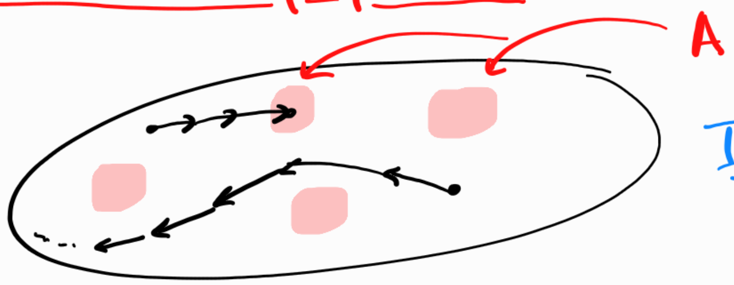
Then I meets every component of G. 

$\rightsquigarrow \exists$ a Borel I-one-ended subforest

$\Rightarrow \chi_B(G-I) \leq d$. Fix a Borel d-coloring c of $G-I$ and extend it to I by giving each $x \in I$ the least color not used by c on any nbr of x.

$< d$

Proof of the proposition:



Idea: Run away from A!.. if you can.

For $x \in V(G)$, let $\underbrace{\text{dist}}_{\text{graph distance in } G}(x, A) :=$ min length of a path in G from x to a vtx in A.

$\text{dist}(x, A) < \infty$ for all $x \in V(G)$.

Say a path $x_0, x_1, x_2, \ldots$ in G is A-repulsive if $\text{dist}(x_{n+1}, A) > \text{dist}(x_n, A)$ $\forall n$.

$R := \{x \in A^c :$ $\exists$ an infinite A-repulsive path starting from $x\}$.

The set R is Borel because G is locally finite (König's lemma again):

For $L \in \mathbb{N}$, let $R_L := \{x \in A^c : \exists$ an A-repulsive path of length L starting from $x\}$.

Borel by Luzin-Novikov.

Clearly, $R \subseteq \bigcap_{L \in \mathbb{N}} R_L$. But in fact,

since G is locally finite, $R = \bigcap_{L \in \mathbb{N}} R_L$.

(idea: If $x \in \bigcap_{L \in \mathbb{N}} R_L$, then x has a nbr y s.t.

$\mathrm{dist}(y, A) > \mathrm{dist}(x, A)$ and $y \in \bigcap_{L \in \mathbb{N}} R_L$)

details: exercise!

TBC