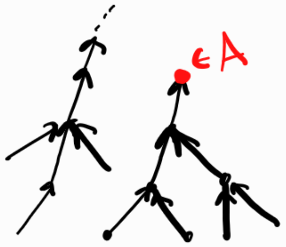
Math 223D, Lecture 24

F is one-ended if there is no infinite F-descending sequence

$$x_0 \xleftarrow{F} x_1 \xleftarrow{F} x_2 \xleftarrow{F} x_3 \longleftarrow \cdots\cdots$$

G, $A \subseteq V(G)$: A-one-ended subforest of G is a one-ended function $F: A^c \to V(G)$ s.t. $F(x)$ is a nbr of x for all $x \in A^c$.



Proposition: G locally finite Borel graph, $A \subseteq V(G)$ a Borel set that meets every component of G, then there exists a Borel A-one-ended subforest.

Pf: $\operatorname{dist}(x, A)$ = graph distance from x to A

= min # of edges in a path in G from x to A.

A path $x_0, x_1, \ldots$ in G is A-repulsive if

$$\operatorname{dist}(x_{n+1}, A) > \operatorname{dist}(x_n, A) \quad \forall n.$$

$R := \{x \in A^c :$ there is an infinite A-repulsive path starting at $x\}$.

Last time: R is Borel. ← this uses that G is locally finite

Note: $x \in R \Rightarrow x$ has a nbr $y \in R$ s.t. $\operatorname{dist}(y, A) > \operatorname{dist}(x, A)$

$x \in A^c \setminus R \Rightarrow x$ has a nbr y s.t. $\operatorname{dist}(y, A) < \operatorname{dist}(x, A)$

BIG idea #2: Borel choice for countable sets!

Thm: X, Y standard Borel, $A \subseteq X \times Y$ a Borel set.

Assume that for each $x \in X$, the fiber A_x is ctbl and $\neq \emptyset$

Suppose that for all $x \in X$, the fiber A_x is ctbl and $\neq \emptyset$.
Then there is a Borel function $f : X \to Y$ s.t. $\underbrace{f(x) \in A_x}$ $\forall x \in X$.

f "chooses" a pt in A_x for each x

Proof: Apply Luzin–Novikov: Borel partial maps $f_n : X \dashrightarrow Y$ s.t. $\forall x \in X$, $A_x = \{ f_n(x) : x \in \mathrm{dom}(f_n) \}$.
In particular, $\underline{f_n(x) \in A_x}$ for all $\underbrace{x \in \mathrm{dom}(f_n)}$.

Define $N(x) := \min \{ n \in \mathbb{N} : x \in \mathrm{dom}(f_n) \}$ and

let $f(x) := f_{N(x)}(x)$. ☺

Using this, we obtain a Borel function $F : A^c \to V(G)$ s.t.:

$x \in R \Rightarrow F(x)$ is a nbr of x s.t. $F(x) \in R$ and
$\mathrm{dist}(F(x), A) > \mathrm{dist}(x, A)$

$x \in A^c \setminus R \Rightarrow F(x)$ is a nbr of x s.t. $\mathrm{dist}(F(x), A) < \mathrm{dist}(x, A)$.

We claim that F is one-ended, as desired! ☺

Suppose not: $x_0, x_1, x_2, \ldots$ is an F-descending sequence.

Case ①: All x_n are in R: $\Rightarrow$ $\mathrm{dist}(x_n, A) > \mathrm{dist}(x_{n+1}, A)$ for all $n \in \mathbb{N}$

$$x_0 \xleftarrow{F} x_1 \xleftarrow{F} x_2 \xleftarrow{F} x_3 \xleftarrow{F} \cdots$$

$\mathrm{dist}(x_0, A) > \mathrm{dist}(x_1, A) > \mathrm{dist}(x_2, A) > \cdots$ IMPOSSIBLE!

Case ②: Some x_n is not in R $\Rightarrow$ $x_m \notin R$ $\forall m \geq n$, so

$$x_n \xleftarrow{F} x_{n+1} \xleftarrow{F} x_{n+2} \leftarrow \cdots$$

$\cancel{A}$ R $\quad$ $\cancel{A}$ R $\quad$ $\cancel{A}$ R

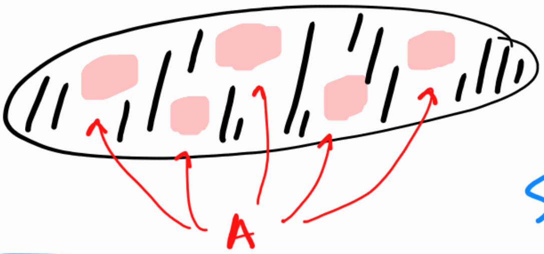
$\mathrm{dist}(x_n, A) < \mathrm{dist}(x_{n+1}, A) < \mathrm{dist}(x_{n+2}, A) < \cdots$

$\Rightarrow x_n, x_{n+1}, x_{n+2}, \dots$ is an A-repulsive infinite path

$\Rightarrow x_n \in R$, a contradiction! ∎

Corollary: G a Borel graph of max. deg. $d \in \mathbb{N}$, $A \subseteq V(G)$ a Borel set that meets every component of G, then $\chi_B(G - A) \leq d$.

G



Hope: working with $G[A]$ is easier than working with G.

Specifically, we'll choose A so that every component of $G[A]$ is finite

all components are finite

BIG idea #3: Component-finite graphs are easy (from the pt of view of Borel combinatorics)

E.g.

Thm: Let G be a Borel component-finite graph. Then $\boxed{\chi_B(G) = \chi(G).}$

Proof: Let $V_n := \{x \in V(G) : \text{the component of } x \text{ has } n \text{ vertices}\}$
$=$ the union of all components of size n

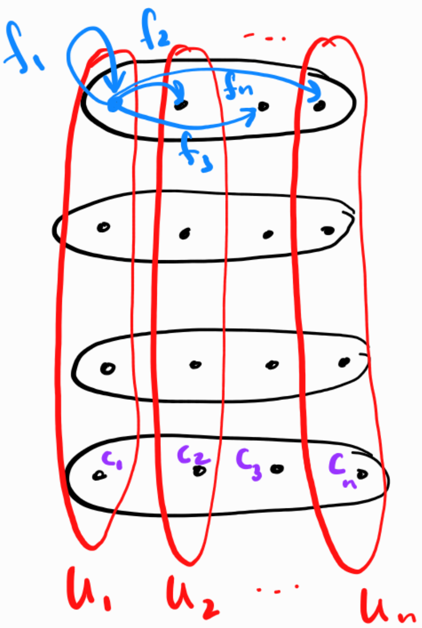
This is a Borel set by Luzin-Novikov (exercise; hint: the relation "x and y are in the same component" is Borel)

$V(G) = V_1 \cup V_2 \cup \dots$

No edges between V_i and V_j for $i \neq j$, so we may color each V_i separately.

So fix $n \in \mathbb{N}$ and pass to $G[V_n]$

$\rightsquigarrow$ from now on, assume that every component of G has exactly n vertices.



Let $<$ be a Borel linear order on $V(G)$. For $1 \le i \le n$, let

$U_i := \{x : x$ is the i^{th} vtx of its component according to $<\}$

Borel by LN

Also, let $f_i(x)$ be the i^{th} vtx in the component of x

(i.e., $f_i(x)$ is the unique vtx from U_i in the same component as x)

Let $k := \chi(G)$. Want: find a Borel k-coloring.

$$A := \{ (x, c) \in U_1 \times \{0, \dots, k-1\}^n :$$
$$\forall\, 1 \le i < j \le n : \{f_i(x), f_j(x)\} \in E(G) \Rightarrow$$
$$\Rightarrow c_i \neq c_j \}$$

$\chi(G) = k$

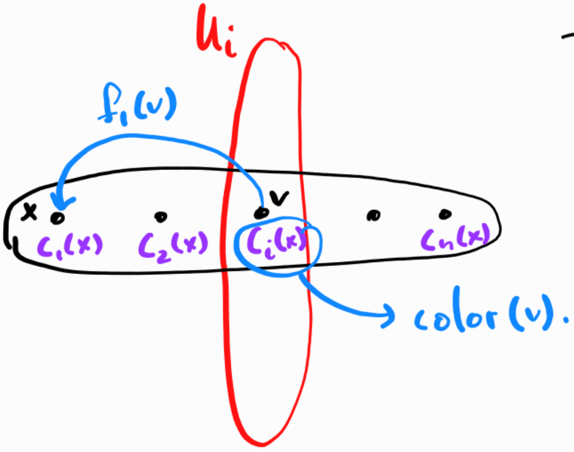
For each $x \in U_1$, the fiber A_x is finite and $\neq \emptyset$

$\Rightarrow$ there is a Borel function $c : U_1 \to \{0, \dots, k-1\}^n$ s.t. $c(x) \in A_x$ for all $x \in U_1$.

$c = (c_1, \dots, c_n)$, $c_i : U_1 \to \{0, \dots, k-1\}$

Define, for $v \in V(G)$,

$$\text{color}(v) := c_i(f_1(v)) \text{ where } v \in U_i.$$



This is a proper Borel k-coloring, as desired!

How to extend the coloring to A?

Different sets of available colors to different vertices...

BIG idea #4: Use fancy results from finite combinatorics/graph theory! ☺