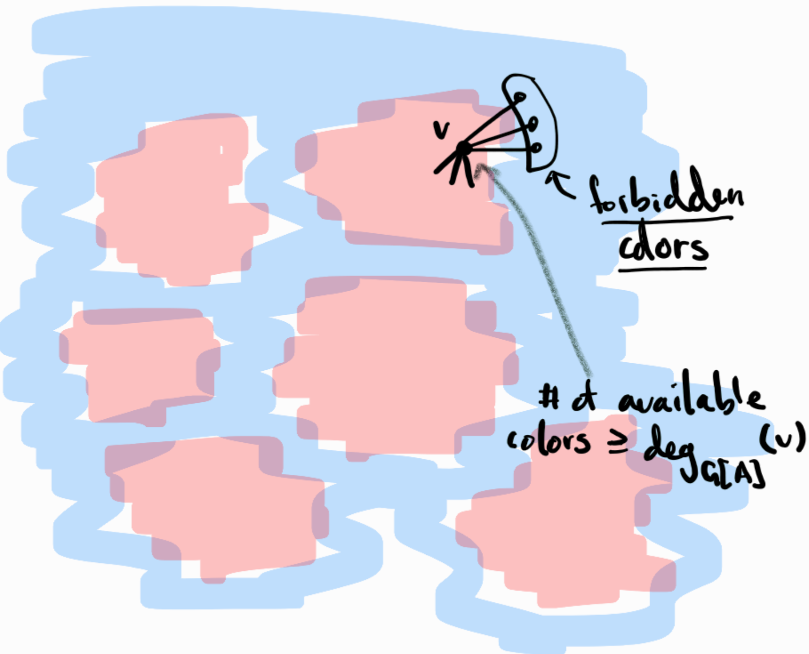
Math 223D, Lecture 25

Final Presentations: Fill out the poll!

Procedure: 48 hours in advance, get ~5 problems from HW

At the exam, I will choose 3 of them for you to present. You will have ≤ 7 minutes per problem

You should outline the main idea of the solution (I may ask for details)



G Borel graph, max deg $d \in \mathbb{N}$

Want: a Borel d-coloring

Plan:

1. Find a "suitable" Borel set $A \subseteq V(G)$ that meets every component of G and s.t. $G[A]$ is component-finite
2. Find a Borel d-coloring of $G - A$ ✓
3. Extend the coloring to A.

↳ ? (Use tools from classical graph theory!)

Vizing / Erdős–Rubin–Taylor 70s

Defn: G a graph. A list assignment for G: a function

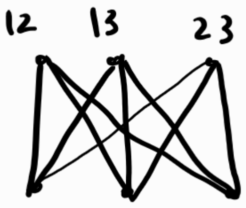
$$L : V(G) \to [\mathbb{N}]^{<\infty}. \quad L(v) = \text{"available colors for } v\text{"}$$

An L-coloring: a map $c : V(G) \to \mathbb{N}$ s.t. $c(v) \in L(v)$ for all $v \in V(G)$.

Proper if it's proper ¯_(ツ)_/¯ $c(u) \neq c(v)$ for all $uv \in E(G)$.

L-colorable: has a proper L-coloring.

Example:



← not L-colorable

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Exercise: There exists a finite bipartite graph G with a list assignment L s.t. $|L(v)| > 10^{10^{10}}$ for all v, but G is not L-colorable

(or anything)

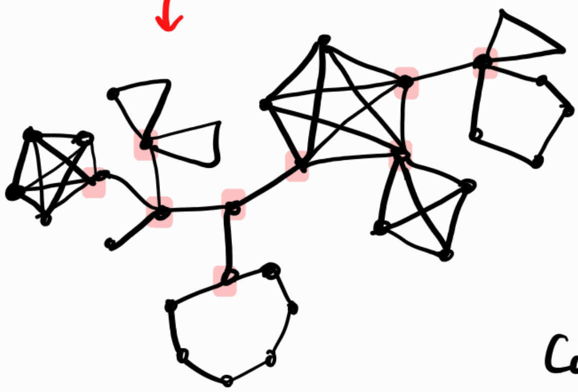
Defn: G is degree-list-colorable if G is L-colorable for every list assignment L s.t. $\underbrace{|L(v)| \geq \deg_G(v) \text{ for all } v \in V(G)}_{\text{"degree-list assignment"}}$

Note (exercise): Every loc. finite graph G is L-colorable when $|L(v)| > \deg_G(v)$ for all $v \in V(G)$ (see HW 7 for Borel versions)

Question: Which graphs are degree-list-colorable?

Thm (Borodin / Erdős–Rubin–Taylor):
A connected graph (locally finite) G is degree-list-colorable unless G is a Gallai tree: Every block is a clique (including K_2) or an odd cycle

example

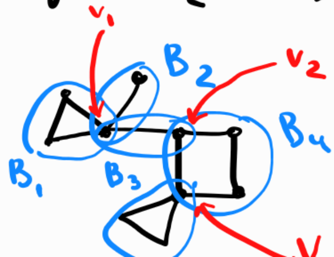


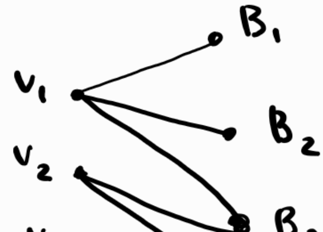
"built" from cliques & odd cycles by "joining" them at vertices in a "tree-like" fashion

Can form the block-cutpoint tree:

vertices = blocks $\cup$ cut points

edges = $\{\{B, v\} :$ the cutpt v is in the block $B\}$





Fact (exercise): A connected loc. finite graph G is a Gallai tree $\iff$ all connected finite induced subgraphs of G are Gallai trees.

Thm: Let G be a Borel graph of max deg $d \in \mathbb{N}$. Suppose no component of G is a d-regular Gallai tree. Then $\chi_B(G) \le d$.

Proof: Let $V_0 := \{v \in V(G) : \text{the component of } v \text{ is not } d\text{-regular}\}$
$V_1 := \{v \in V(G) : \text{the comp. of } v \text{ is } d\text{-regular}\}$.

Note: V_0, V_1 are Borel by Luzin-Novikov.

We know that $\chi_B(G[V_0]) \le d$ (see Lecture 23)

It remains to show $\chi_B(G[V_1]) \le d$. May assume $V_1 = V(G)$.

$\Rightarrow$ G is d-regular. So no component of G is a Gallai tree.

$\Rightarrow$ Every component has a finite subset F s.t. $G[F]$ is connected & not a Gallai tree

BIG idea #5: Maximal disjoint families!

Prop: Let $\mathcal{H} \subseteq [X]^{<\infty}$ be a Borel family of finite sets.
$\uparrow$ st. Borel

If each $A \in \mathcal{H}$ intersects ctbly many other sets in $\mathcal{H}$, then there exists a Borel maximal disjoint subfamily $\mathcal{H}' \subseteq \mathcal{H}$.

Why? The intersection graph $G_{\mathcal{H}}$ is locally ctbl

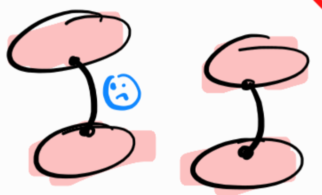
$\Rightarrow$ $\chi_B(G_{\mathcal{H}}) \le \aleph_0$ (Generalized Feldman-Moore, see Homework 6)

$\Rightarrow G_{\mathcal{H}}$ has a Borel maximal indep. set $\mathcal{H}' \subseteq \mathcal{H}$

Now, define $\mathcal{H} \subseteq [V(G)]^{<\omega}$:

$$\mathcal{H} := \{ F \in [V(G)]^{<\omega} : G[F] \text{ is connected and not a Gallai tree} \}$$

$\Rightarrow$ Get a Borel maximal disjoint subfamily $\mathcal{H}' \subseteq \mathcal{H}$.

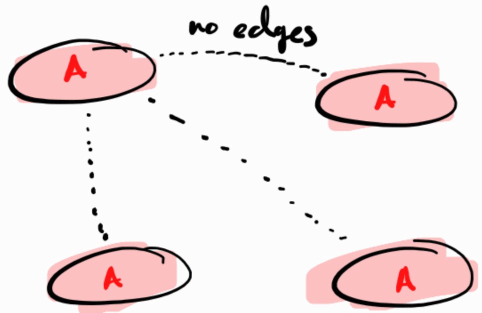


Note: Every component of G contains a set from $\mathcal{H}'$.

Define a graph Γ: vertices $= \mathcal{H}'$, edges $= \{ \{F, F'\} \in [\mathcal{H}']^2 :$ there is an edge from F to F' in $G \}$.

Γ is locally finite, so

$\chi_B(\Gamma) \leq \aleph_0 \Rightarrow \Gamma$ has a Borel maximal indep. set $\mathcal{H}'' \subseteq \mathcal{H}'$



Define: $A := \bigcup \mathcal{H}''$

$= \{ v : \exists F \in \mathcal{H}'' \text{ s.t. } v \in F \}$

A is Borel because such F must be unique.

$\mathcal{H}''$ = the connected components of $G[A]$

Not Gallai trees

A meets every comp. of $G \Rightarrow$ Find a Borel proper d-coloring c of $G - A$.

For $v \in A$ define $L(v) := \{ 0 \leq i < d : i \neq c(u)$ for any nbr $u \in A^c$ of $v\}$.

Note: $|L(v)| \geq \deg_{G[A]}(v)$

Since no component of $G[A]$ is a Gallai tree, $G[A]$ is degree-list colorable, so it has a proper L-coloring. As $G[A]$ is component-finite, it also has a Borel proper L-coloring $c' : A \to \{0, \ldots, d-1\}$ (see HW 7).

Finally, $c \cup c'$ is a Borel proper d-coloring of G! ☺

What to do with components that are Gallai trees?

We'll argue they are measurably d-colorable.

Key tool: Let G be a Borel forest s.t.

can extend this to "Gallai forests"

1. no vertices of degree < 2
2. no ∞ path of vertices of degree 2
3. max degree $d \in \mathbb{N}$

Then, for any prob. measure μ on $V(G)$, there exist a μ-null Borel set $A \subseteq V(G)$ and a Borel A-one-ended subforest $F : A^c \to V(G)$ of G.